The Random Wave Conjecture and Arithmetic Quantum Chaos

Peter Humphries

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Classical Mechanics on a Surface M

Let M be a compact Riemannian surface.

State of Motion: a possible position $x \in M$ and momentum $\xi \in S_x^* M$.

Phase Space: set of all possible states of motion $S^* M$.

Classical Dynamics: motion of a particle over time as governed by Newton's laws, that is, via the geodesic flow $g_t : S^* M \to S^* M$.

Quantum Mechanics on a Surface M

State of Motion: a wave function $\phi : M \times [0, \infty) \to \mathbb{C}$ of position and time, so that the probability of a particle with wave duality ϕ being in a region $B \subset M$ at time t is

$$\frac{\int_B |\phi(x,t)|^2 \, d\mathrm{vol}(x)}{\int_M |\phi(x,t)|^2 \, d\mathrm{vol}(x)}.$$

Space of States: Hilbert space $L^2(M, d\mathrm{vol})$ of square-integrable functions.

Quantum Dynamics: evolution of a wave function over time as governed by Schrödinger's equation

$$i\hbar \frac{\partial \phi}{\partial t} = -\Delta \phi.$$

Stationary States

Stationary states are normalised states ϕ such that the probability densities $|\phi(x,t)|^2 \, d\mathrm{vol}(x)$ are independent of time, so that there exists some $\lambda \in \mathbb{R}$ such that

$$\phi(x,t) = \phi(x) e^{-\frac{i\lambda}{\hbar} t}.$$

These correspond to solutions of the eigenvalue problem

$$\Delta\phi + \lambda\phi = 0.$$

Stationary States

Theorem

Let M be a compact Riemannian manifold, and consider the eigenvalue problem $\Delta\phi + \lambda\phi = 0$ on M. There exists a discrete spectrum of eigenvalues $0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ tending to infinity, and a corresponding sequence of eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ forming a complete orthonormal basis of $L^2(M, d\mathrm{vol})$.

Thus all states can be represented as linear combinations of stationary states.

Semiclassical Limit

Heuristically, the classical model should appear as the limiting behaviour of the quantum model: we expect the limiting nature of the quantum dynamics to be similar to that of the classical dynamics.

Mathematically, we wish to show that the limiting behaviour of the Laplacian eigenfunctions $\{\phi_j\}$ somehow reflects the properties of the geodesic flow.

Conjecture (Berry (1977))

Geodesic flow on S^*M is ergodic $\implies$ Laplacian eigenfunctions behave randomly in the large eigenvalue limit.

Equidistribution

Let M be a topological space and μ a probability measure on M. Let $\{\mu_T\}$ be a family of probability measures on M.

Definition

The family of probability measures $\{\mu_T\}$ equidistribute on M w.r.t. μ if

$$\lim_{T \to \infty} \mu_T(B) = \mu(B)$$

for every continuity set $B \subset M$ (boundary has μ-measure zero).

Equidistribution

Let M be a topological space and μ a probability measure on M. Let $\{\mu_T\}$ be a family of probability measures on M.

Definition

The family of probability measures $\{\mu_T\}$ equidistribute on M w.r.t. μ if

$$\lim_{T\to\infty} \int_M f(x)\, d\mu_T(x) = \int_M f(x)\, d\mu(x)$$

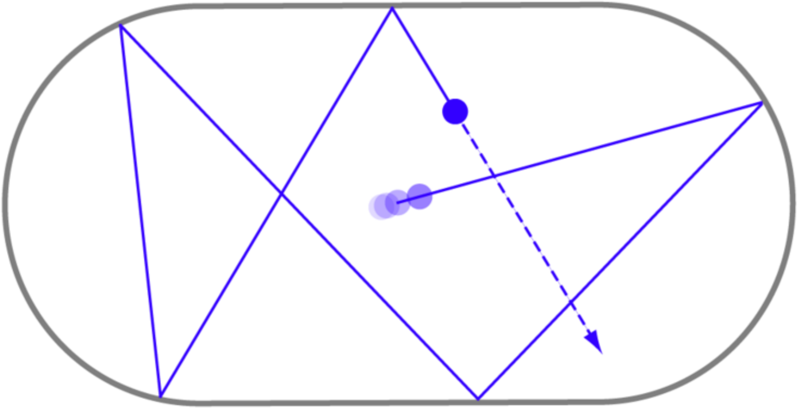
for all $f \in C_b(M)$ (continuous bounded).

Classical Ergodicity

Definition

A classical system is *ergodic* if the orbit of almost every $(x, \xi) \in S^*M$ under the classical dynamics of geodesic flow is equidistributed in S^*M.

Classical Dynamics on the Bunimovich Stadium



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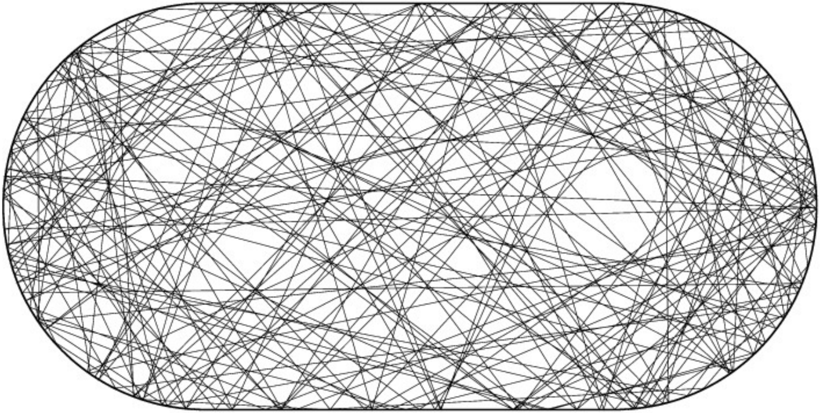
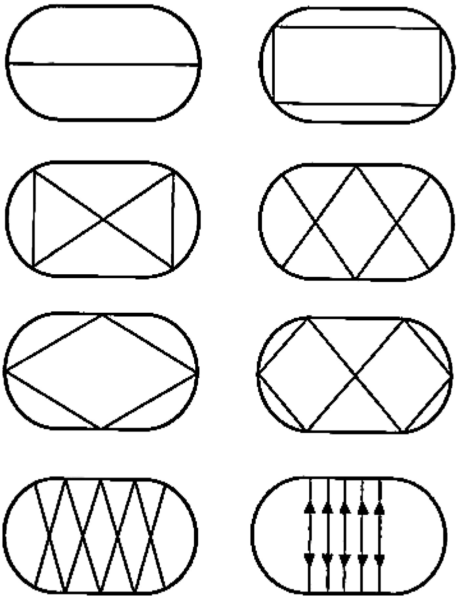


Classical Dynamics on the Bunimovich Stadium



Definition

A quantum system is *(uniquely) ergodic* if the probability densities $|\phi_j(x)|^2$ of the stationary states become equidistributed as $\lambda_j \to \infty$.

That is, for all continuity sets $B \subset M$,

$$\frac{\int_B |\phi_j(x)|^2 \, d\mathrm{vol}(x)}{\int_M |\phi_j(x)|^2 \, d\mathrm{vol}(x)} \to \frac{\mathrm{vol}(B)}{\mathrm{vol}(M)} \quad \text{as } \lambda_j \to \infty.$$

Equivalently, the probability measures $d\mu_j(x) = |\phi_j(x)|^2 \, d\mathrm{vol}(x)$ converge in the weak-$*$ topology to the normalised uniform volume measure $d\mu = d\mathrm{vol}$.

Quantum Unique Ergodicity

Conjecture (QUE on Configuration Space)

Let M be a compact Riemannian surface whose geodesic flow is ergodic. Then the probability densities $|\phi_j(x)|^2$ of the stationary states become equidistributed as $\lambda_j \to \infty$.

This is QUE on configuration space. Stronger formulation is QUE on phase space: probability measures $|\phi_j(x)|^2 \, d\mathrm{vol}(x)$ replaced with microlocal lifts on S^*M; equidistribution w.r.t. Liouville measure.

Quantum Ergodicity

Theorem (Shnirelman (1974), Colin de Verdière (1985), Zelditch (1987))

Let M be a compact Riemannian manifold whose geodesic flow is ergodic. Then there exists a subsequence $\{j_k\}$ of density 1 of the probability densities $|\phi_j(x)|^2$ of the stationary states that become equidistributed as $\lambda_{j_k} \to \infty$.

A subsequence $\{j_k\}$ has density $\alpha \in [0, 1]$ if

$$\frac{\#\{k : j_k \leq N\}}{N} \to \alpha \quad \text{as } N \to \infty$$

Berry's Random Wave Model

QE and QUE are manifestations of Berry's random wave conjecture.

Conjecture (Berry (1977))

As $\lambda_j \to \infty$, Laplacian eigenfunctions on surfaces with chaotic classical dynamics are well-modelled by random waves.

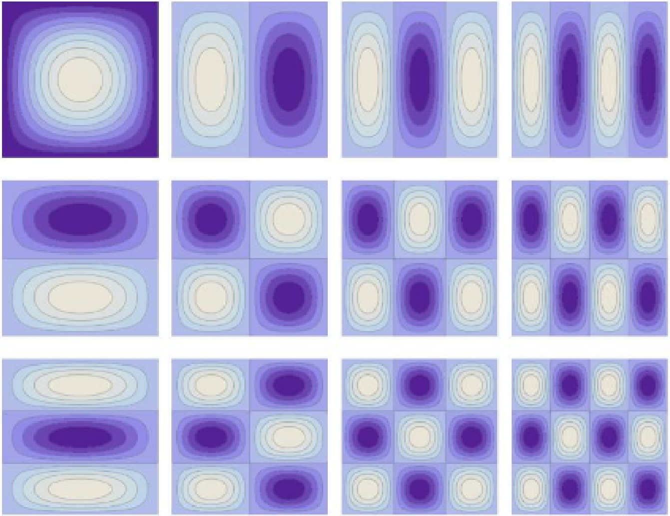
Random waves are functions of the form

$$\sum_{j \in J} a_j \phi_j,$$

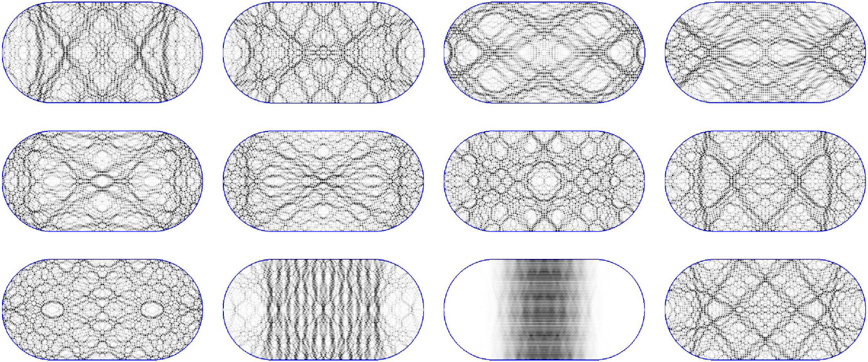
where $\{\phi_j\}$ is an orthonormal basis of Laplacian eigenfunctions, $\{a_j\}$ are i.i.d. Gaussian random variables, and $J \subset \mathbb{N}$.

This model allows us to study Laplacian eigenfunctions in the large eigenvalue limit *probabilistically* (see work of de Courcy-Ireland, Han–Tacy).

Semiclassical Limit on the Torus



Semiclassical Limit on the Bunimovich Stadium



Counterexample to QUE: the Bunimovich Stadium

Theorem (Hassell (2010))

For the Bunimovich stadium, there exist subsequences $\{j_k\}$ of density 0 of the probability densities $|\phi_j(x)|^2$ of the stationary states that scar in certain regions as $\lambda_{j_k} \to \infty$.

Refinement of Quantum Unique Ergodicity

Conjecture (Rudnick–Sarnak (1994))

Let M be a compact Riemannian surface of negative sectional curvature*. Then the probability densities $|\phi_j(x)|^2$ of the stationary states become equidistributed as $\lambda_j \to \infty$.*

Negative sectional curvature implies that the geodesic flow is ergodic (and much more).

Conjecture is completely open except for **arithmetic** surfaces.

Rate of Equidistribution: Decay of Error Term

What is the rate of equidistribution of μ_j on M w.r.t. μ?

Goal

Find the most rapidly decreasing function $\alpha(j)$ for which

$$\mu_j(B) = \mathrm{vol}(B) + O_B(\alpha(j))$$

for a fixed continuity set $B \subset M$.

Informally, determine how quickly the L^2-mass of Laplacian eigenfunctions spread out randomly on M.

Rate of Equidistribution: Decay of Error Term

Heuristic

Like for *random waves*, we should expect square-root cancellation: since the Planck scale is $\hbar = \lambda_j^{-1/2}$, we should hope for $\alpha(j) \approx \lambda_j^{-1/4}$.

Conjecture (QUE at an optimal rate)

Let M be a compact Riemannian surface of negative sectional curvature. Then the probability densities $|\phi_j(x)|^2$ of the stationary states equidistribute on any fixed ball $B_R(y) \subset M$ as $\lambda_j \to \infty$ with an error term of size $O_\varepsilon(\lambda_j^{-1/4+\varepsilon})$.

Best known result: **QE** at a rate $O((\log \lambda_j)^{-1})$ (Zelditch).

Rate of Equidistribution: Small Scale Equidistribution

What is the rate of equidistribution of μ_j on M w.r.t. μ?

Goal

Find the most rapidly decreasing function $\alpha(j)$ for which

$$\frac{\mu_j(B_{R_j}(y))}{\mu(B_{R_j}(y))} \to 1$$

for a sequence of radii $R_j = \alpha(j)$.

Informally, determine the scale at which Laplacian eigenfunctions no longer look random.

How small does a ball have to be to **not** contain the expected amount of L^2-mass of a Laplacian eigenfunction?

Rate of Equidistribution: Small Scale Equidistribution

Heuristic

Like for *random waves*, we should expect small scale equidistribution provided we are at a scale above the Planck scale $\hbar = \lambda_j^{-1/2}$.

Conjecture (Planck scale QUE)

Let M be a compact Riemannian surface of negative sectional curvature. Then the probability densities $|\phi_j(x)|^2$ of the stationary states equidistribute on shrinking balls $B_{R_j}(y)$ with fixed centre as $\lambda_j \to \infty$ provided that $R_j \gg \hbar^{1-\delta} = \lambda_j^{-\frac{1-\delta}{2}}$ for some fixed $\delta > 0$.

Best known result: **QE** at a rate $O((\log \lambda_j)^{-\delta})$ for some small $\delta > 0$ (Han, Hezari–Rivière).

Rate of Equidistribution

One can think of small scale equidistribution in terms of random variables. Define $X_{j;R_j} : M \to [0, \infty)$ by

$$X_{j;R_j}(y) := \frac{\mu_j(B_{R_j}(y))}{\mu(B_{R_j}(y))},$$

where R_j shrinks at some rate $\alpha(j)$ as $j \to \infty$.

- Equidistribution implies this has expectation 1.
- Small scale equidistribution is the pointwise convergence of this random variable to 1.
- Small scale equidistribution almost everywhere

$$\mu\left(\left\{y \in M : \left|\frac{\mu_j(B_{R_j}(y))}{\mu(B_{R_j}(y))} - 1\right| > \varepsilon\right\}\right) \to 0$$

is convergence in probability of this random variable to 1.

Gaussian Moments Conjecture

Heuristic

Like *random waves*, Laplacian eigenfunctions should exhibit Gaussian random behaviour in the large eigenvalue limit.

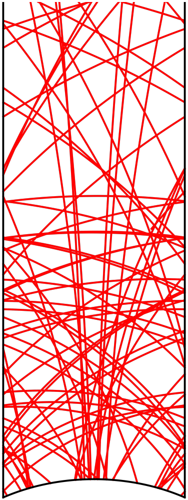
Conjecture (Gaussian Moments Conjecture)

For every nonnegative integer n,

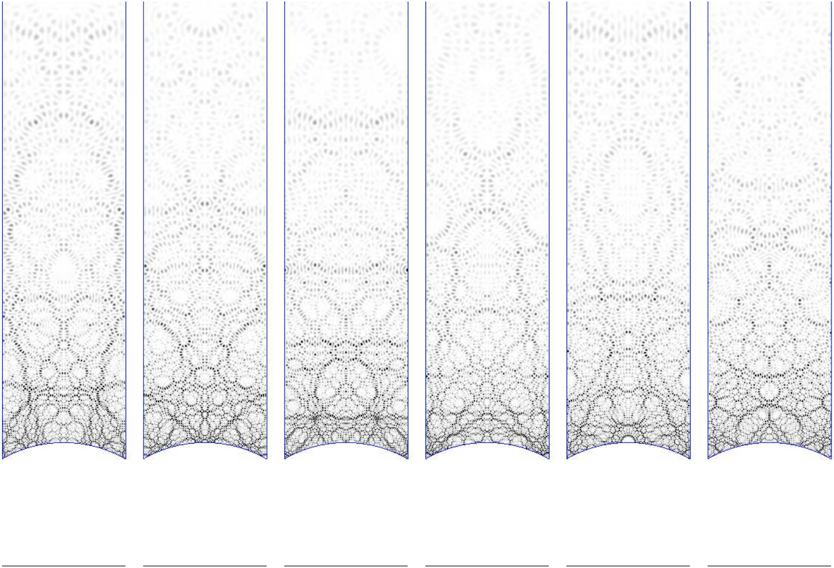
$$\lim_{j\to\infty}\int_M \phi_j(x)^n \, d\mathrm{vol}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}\, dx$$
$$= \begin{cases} \dfrac{2^{n/2}}{\sqrt{\pi}}\Gamma\left(\dfrac{n+1}{2}\right) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Sogge gives upper bounds. Trivial for $n \in \{0, 1, 2\}$; unknown otherwise... unless M is arithmetic.

Geodesic Flow on $\Gamma\backslash\mathbb{H}$



Eigenfunctions of the Laplacian on $\Gamma\backslash\mathbb{H}$



Eigenfunctions of the Laplacian on $\Gamma\backslash\mathbb{H}$

$M = \Gamma\backslash\mathbb{H}$: Laplacian eigenfunctions are smooth functions $\phi : \mathbb{H} \to \mathbb{C}$ satisfying

- $\phi\left(\frac{az+b}{cz+d}\right) = \phi(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma := \mathrm{SL}_2(\mathbb{Z})$,
- $\Delta\phi(z) := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(z) = -\lambda_\phi \phi(z)$ for some $\lambda_\phi \geq 0$.

$\Gamma\backslash\mathbb{H}$ has constant negative curvature (which implies geodesic flow is ergodic), is noncompact, but of finite volume: $\mathrm{vol}(\Gamma\backslash\mathbb{H}) = 1$ with respect to the measure $d\mathrm{vol}(z) = d\mu(z) := \frac{3}{\pi} y^{-2}\, dx\, dy$.

Spectral Decomposition of $\Gamma\backslash\mathbb{H}$

$\Gamma\backslash\mathbb{H}$ not compact $\implies$ spectrum of the Laplacian not discrete.
L^2-spectral decomposition is

$$f(z) = \langle f, 1\rangle + \sum_{\psi\in\mathcal{B}} \langle f, \psi\rangle \psi(z) \\ + \frac{1}{12}\int_{-\infty}^{\infty} \left\langle f, E\left(\cdot, \frac{1}{2} + it\right)\right\rangle E\left(z, \frac{1}{2} + it\right)\, dt.$$

Converges uniformly for $f \in C_c^\infty(\Gamma\backslash\mathbb{H})$.

- $\mathcal{B}$ orthonormal basis of Hecke–Maaß cusp forms / nonconstant Laplacian eigenfunctions (complicated number theoretically),
- $E(z, 1/2 + it)$ Eisenstein series / generalised eigenfunction (complicated analytically), with Laplacian eigenvalue $\lambda = 1/4 + t^2$,
- $\langle \psi_1, \psi_2\rangle := \int_{\Gamma\backslash\mathbb{H}} \psi_1(z)\overline{\psi_2(z)}\, d\mu(z)$.

Quantum Unique Ergodicity for $\Gamma\backslash\mathbb{H}$

Theorem (Lindenstrauss (2006), Soundararajan (2010))

For $\phi_j \in \mathcal{B}$ with Laplacian eigenvalue $\lambda_j = 1/4 + t_j^2$,

$$\lim_{t_j \to \infty} \int_B |\phi_j(z)|^2 \, d\mu(z) = \mathrm{vol}(B)$$

for every continuity set $B \subset \Gamma\backslash\mathbb{H}$.

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For $\phi_j \in \mathcal{B}$ with Laplacian eigenvalue $\lambda_j = 1/4 + t_j^2$,

$$\lim_{t_j \to \infty} \int_{\Gamma\backslash\mathbb{H}} f(z) |\phi_j(z)|^2 \, d\mu(z) = \int_{\Gamma\backslash\mathbb{H}} f(z) \, d\mu(z)$$

for all $f \in C_b(\Gamma\backslash\mathbb{H})$.

Theorem (Luo–Sarnak (1995))

For $E(z, 1/2 + it)$ with Laplacian eigenvalue $\lambda = 1/4 + t^2$,

$$\lim_{t\to\infty} \frac{1}{\log\lambda} \int_B \left| E\left(z, \frac{1}{2} + it\right) \right|^2 d\mu(z) = \mathrm{vol}(B)$$

for every compact continuity set $B \subset \Gamma\backslash\mathbb{H}$.

Quantum Unique Ergodicity for $\Gamma\backslash\mathbb{H}$

Theorem (Luo–Sarnak (1995))

For $E(z, 1/2 + it)$ *with Laplacian eigenvalue* $\lambda = 1/4 + t^2$*,*

$$\lim_{t\to\infty} \frac{1}{\log\lambda} \int_{\Gamma\backslash\mathbb{H}} f(z) \left| E\left(z, \frac{1}{2} + it\right)\right|^2 d\mu(z) = \int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu(z)$$

for all $f \in C_c(\Gamma\backslash\mathbb{H})$*.*

Quantum Unique Ergodicity for $\Gamma\backslash\mathbb{H}$

Strategy of Proof.

QUE needs to be proven for every function $f \in C_b(\Gamma\backslash\mathbb{H})$. Spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$ allows us to approximate f by linear combinations of the constant function, Laplacian eigenfunctions $\psi \in \mathcal{B}$, and direct integrals of Eisenstein series $E(z, 1/2 + it)$.

This reduces QUE to showing that for $\psi \in \mathcal{B}$ and $t \in \mathbb{R}$,

$$\int_{\Gamma\backslash\mathbb{H}} \psi(z)|\phi_j(z)|^2 \, d\mu(z) = \langle \psi, |\phi_j|^2 \rangle \to 0,$$

$$\int_{\Gamma\backslash\mathbb{H}} E\left(z, \frac{1}{2} + it\right) |\phi_j(z)|^2 \, d\mu(z) = \left\langle E\left(\cdot, \frac{1}{2} + it\right), |\phi_j|^2 \right\rangle \to 0$$

as $t_j \to \infty$.

Similar process with $E(z, 1/2 + it)$ in place of ϕ_j, with additional care for integrating triple product of Eisenstein series. □

The Watson–Ichino Formula

Theorem (Watson (2002), Ichino (2008))

For $\psi, \phi_j \in \mathcal{B}$,

$$\langle |\phi_j|^2, \psi \rangle = \mathcal{A}_{\phi_j,\psi} \cdot \mathcal{S}_{\phi_j,\psi},$$

where $\mathcal{A}_{\phi_j,\psi}$ is the arithmetic part and $\mathcal{S}_{\phi_j,\psi}$ is the spectral part.

A similar identity holds when either ψ or ϕ_j is replaced by an Eisenstein series (Rankin–Selberg).

Proof is via representation-theoretic methods.

Relies heavily on the arithmeticity of $\Gamma\backslash\mathbb{H}$, as well as the fact that $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ is a symmetric space.

The Watson–Ichino Formula: the Spectral Part

The spectral part $\mathcal{S}_{\phi_j,\psi}$ depends only on t_j and t_ψ and can be written explicitly in terms of products of the gamma function. By Stirling's formula,

$$|\mathcal{S}_{\phi_j,\psi}|^2 \approx \frac{1}{(1+t_\psi)(|4t_j^2 - t_\psi^2|+1)^{1/2}} \times \begin{cases} 1 & \text{if } 0 < t_\psi < 2t_j, \\ e^{-\pi(t_\psi - 2t_j)} & \text{if } t_\psi \geq 2t_j. \end{cases}$$

- Polynomial decay if ψ oscillates slower than $|\phi_j|^2$: $\frac{t_\psi}{2t_j} \to 0$,
- Constructive interference if ψ oscillates at the same frequency as $|\phi_j|^2$: $t_\psi \asymp 2t_j$,
- Exponential decay if ψ oscillates faster than $|\phi_j|^2$: $\frac{t_\psi}{2t_j} \to \infty$.

The Watson–Ichino Formula: the Arithmetic Part

The arithmetic part is related to L-functions:

$$|\mathcal{A}_{\phi_j,\psi}|^2 = \frac{L(\frac{1}{2},\psi)L(\frac{1}{2},\mathrm{sym}^2\phi_j\otimes\psi)}{L(1,\mathrm{sym}^2\psi)L(1,\mathrm{sym}^2\phi_j)^2}.$$

Denominator is harmless: $t_j^{-\varepsilon} \ll_{\psi,\varepsilon} \cdots \ll_{\psi,\varepsilon} t_j^{\varepsilon}$.

Generalisations of the Riemann zeta function $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$:

$$L(s,\psi) = \prod_p \prod_{k\in\{1,-1\}} \frac{1}{1-\alpha_\psi^k(p)p^{-s}},$$

$$L(s,\mathrm{sym}^2\phi_j\otimes\psi) = \prod_p \prod_{k\in\{1,-1\}} \prod_{\ell\in\{2,0,-2\}} \frac{1}{1-\alpha_\psi^k(p)\alpha_{\phi_j}^\ell(p)p^{-s}},$$

where each $\alpha_\psi(p), \alpha_{\phi_j}(p) \in \mathbb{C}$ has absolute value 1.

Consequences of the Watson–Ichino Formula

Combining the Watson–Ichino formula with the asymptotics for the spectral part yields the following.

Corollary

QUE for $\Gamma\backslash\mathbb{H}$ with rate $O(t_j^{-\delta})$ follows from the bounds

$$L\left(\frac{1}{2}, \mathrm{sym}^2\phi_j \otimes \psi\right) \ll_{t_\psi} t_j^{1-2\delta}.$$

In particular, QUE for $\Gamma\backslash\mathbb{H}$ with an optimal rate $O_\varepsilon(t_j^{-1/2+\varepsilon})$ follows from the bounds

$$L\left(\frac{1}{2}, \mathrm{sym}^2\phi_j \otimes \psi\right) \ll_{t_\psi,\varepsilon} t_j^{\varepsilon}.$$

Optimal! Cannot replace t_j^{ε} with 1.

Subconvexity

Lemma (Convexity bound)

We have that

$$L\left(\frac{1}{2}, \mathrm{sym}^2\phi_j \otimes \psi\right) \ll (t_\psi + 1)^{1/2}(|4t_j^2 - t_\psi^2| + 1)^{1/2} \ll_{t_\psi} t_j.$$

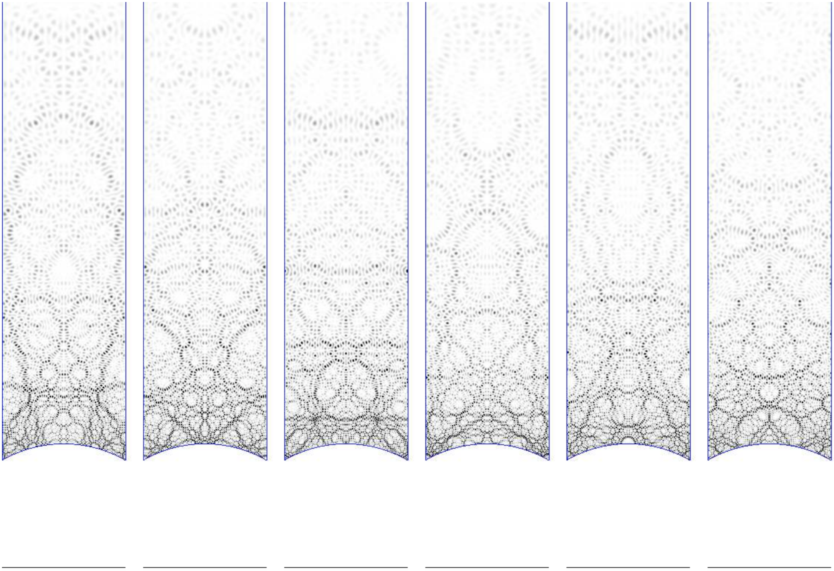
Replacing t_j with $t_j^{1-\delta}$ is known as a *subconvex* bound.
Unknown in this generality!

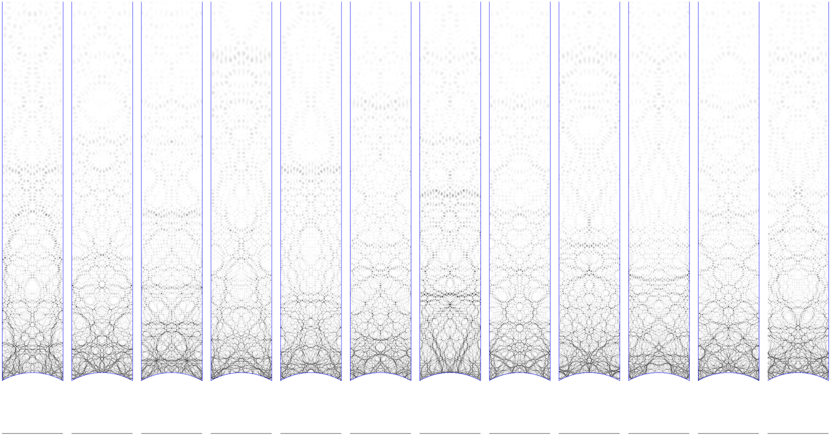
Conjecture (Generalised Lindelöf Hypothesis)

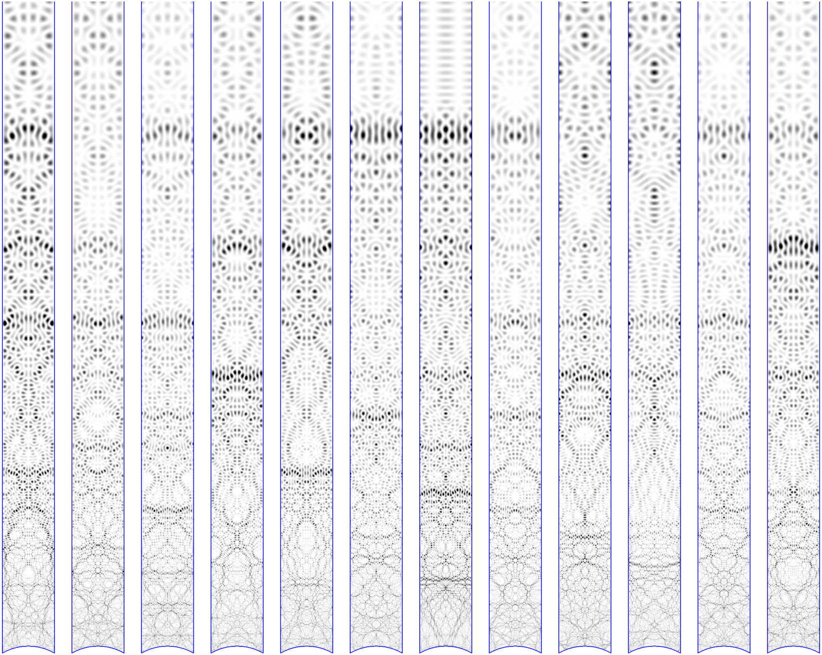
We have that

$$L\left(\frac{1}{2}, \mathrm{sym}^2\phi_j \otimes \psi\right) \ll_\varepsilon (t_\psi + 1)^\varepsilon(|4t_j^2 - t_\psi^2| + 1)^\varepsilon \ll_{t_\psi,\varepsilon} t_j^\varepsilon.$$

Consequence of the generalised Riemann hypothesis. **Optimal!**
Cannot replace t_j^ε with 1.







Planck Scale QUE

Conjecture (Planck scale QUE)

The L^2-mass $|\phi_j|^2$ of Laplacian eigenfunctions ϕ_j on $\Gamma\backslash\mathbb{H}$ equidistribute on shrinking balls $B_R(w)$ with fixed centre $w \in \Gamma\backslash\mathbb{H}$ as $t_j \to \infty$ provided that $R \gg t_j^{-1+\delta}$ for some fixed $\delta > 0$:

$$\lim_{t_j \to \infty} \frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z) = 1.$$

Failure of QUE at the Planck Scale

Theorem (H. (2018))

There exists a countable dense collection of points $w \in \Gamma\backslash\mathbb{H}$ *such that for* $R \ll (\log t_j)^A t_j^{-1}$,

$$\frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z)$$

*does **not** converge to* 1 *as* $t_j \to \infty$.

Small Scale QUE: Conditional Results

Theorem (Young (2016))

For any sequence of $\phi_j \in \mathcal{B}$ and assuming GLH, QUE holds for balls $B_R(w)$ centred at a fixed point w of radius $R \gg t_j^{-\delta}$ with $\delta < 1/3$.

Theorem (H. (2018))

For any sequence of $\phi_j \in \mathcal{B}$ and assuming GLH, QUE holds in almost every ball of radius $R \gg t_j^{-\delta}$ with $\delta < 1$.

Small Scale QUE: Unconditional Results

Theorem (Young (2016))

For $E(z, 1/2 + it)$, QUE holds for balls $B_R(w)$ centred at a fixed point w of radius $R \gg t^{-\delta}$ with $\delta < 1/9$.

Theorem (H. (2018))

For $E(z, 1/2 + it)$, QUE holds in almost every ball of radius $R \gg t^{-\delta}$ with $\delta < 1$.

Theorem (H.–Khan (2020))

For a sequence of dihedral $\phi_j \in \mathcal{B}$, QUE holds in almost every ball of radius $R \gg t_j^{-\delta}$ with $\delta < 1$.

Gaussian Moments Conjecture

Conjecture (Fourth Moment Conjecture)

We have that

$$\lim_{t_j \to \infty} \int_{\Gamma \backslash \mathbb{H}} |\phi_j(z)|^4 \, d\mu(z) = 3.$$

Asymptotics for the Fourth Moment

Theorem (Buttcane–Khan (2017))

The fourth moment conjecture holds for any sequence of $\phi_j \in \mathcal{B}$ under the assumption of GLH.

Theorem (Djanković–Khan (2018))

A regularised version of the fourth moment conjecture holds for $E(z, 1/2 + it)$:

$$\lim_{t \to \infty} \frac{1}{(\log \lambda)^2} \int_{\Gamma \backslash \mathbb{H}}^{\mathrm{reg}} \left| E\left(z, \frac{1}{2} + it\right) \right|^4 \, d\mu(z) \sim 6.$$

Theorem (H.–Khan (2020))

The fourth moment conjecture holds for any sequence of dihedral $\phi_j \in \mathcal{B}$.

Sketch of Proofs

Sketch of proofs:

1. Small scale QUE,
2. Small scale QUE almost everywhere,
3. Asymptotics for the fourth moment.

Small Scale QUE

Small scale QUE for Maaß forms:

$$\frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z) \to 1$$

as $t_j \to \infty$ with $R \gg t_j^{-\delta}$ for $\delta < 1/3$.

Define

$$K_R(z,w) := \begin{cases} \dfrac{1}{\mathrm{vol}(B_R)} & \text{if } \mathrm{dist}(z,w) \leq R, \\ 0 & \text{otherwise.} \end{cases}$$

Parseval's identity gives

$$\frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z) = 1 + \sum_{\psi \in \mathcal{B}} \langle |\phi_j|^2, \psi \rangle \langle \psi, K_R(\cdot, w) \rangle$$
$$+ \frac{1}{12} \int_{-\infty}^{\infty} \left\langle |\phi_j|^2, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle \left\langle E\left(\cdot, \frac{1}{2} + it\right), K_R(\cdot, w) \right\rangle \, dt.$$

Need to show latter two terms are $o(1)$.

The Selberg–Harish-Chandra Transform

Lemma

Let ψ be a Laplacian eigenfunction with eigenvalue $\lambda_\psi = 1/4 + t_\psi^2$. Then

$$\frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} \psi(z)\, d\mu(z) = \langle \psi, K_R(\cdot, w)\rangle = h_R(t_\psi)\psi(w)$$

where the Selberg–Harish-Chandra transform h_R of K_R satisfies

$$h_R(t) = \begin{cases} 1 + o(1) & \text{if } t \ll R^{-1}, \\ O\left(\frac{1}{(Rt)^{3/2}}\right) & \text{if } t \gg R^{-1}. \end{cases}$$

Heuristic

Laplacian eigenfunctions essentially satisfy the mean-value property on balls of radius $R \gg \lambda^{-1/2}$.

Proof uses the fact that $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ is a symmetric space.

Small Scale QUE

Sketch of proof of small scale QUE.

Spectral expansion is

$$\frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z) = 1 + \sum_{\psi \in \mathcal{B}} \langle |\phi_j|^2, \psi \rangle h_R(t_\psi) \psi(w) + \frac{1}{12} \int_{-\infty}^{\infty} \left\langle |\phi_j|^2, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle h_R(t) E\left(w, \frac{1}{2} + it\right) \, dt.$$

Use Watson–Ichino to write $\langle |\phi_j|^2, \psi \rangle = \mathcal{A}_{\phi_j,\psi} \cdot \mathcal{S}_{\phi_j,\psi}$, GLH to bound $\mathcal{A}_{\phi_j,\psi}$, Stirling's approximation to bound $\mathcal{S}_{\phi_j,\psi}$, asymptotics for $h_R(t_\psi)$, and the Weyl law: the latter two terms are $o(1)$ as $t_j \to \infty$ provided that $R \gg t_j^{-\delta}$ with $\delta < 1/3$. □

We fall short of the Planck scale: unable to make use of additional cancellation in the spectral sum.

Small Scale QUE Almost Everywhere

Small scale QUE almost everywhere for Maaß forms:

$$\mathrm{vol}\left(\left\{w \in \Gamma\backslash\mathbb{H} : \left|\frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z) - 1\right| > \varepsilon\right\}\right) \to 0$$

as $t_j \to \infty$ with $R \gg t_j^{-\delta}$ for $\delta < 1$.

Implied by $\mathrm{Var}(\phi_j; R) \to 0$, where

$$\mathrm{Var}(\phi_j; R) := \int_{\Gamma\backslash\mathbb{H}} \left(\frac{1}{\mathrm{vol}(B_R)} \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z) - 1\right)^2 d\mu(w).$$

Spectral Expansion of the Variance

For the variance, we subtract off the main term 1, square, and integrate over $w \in \Gamma\backslash\mathbb{H}$. Parseval then gives

$$\mathrm{Var}(\phi_j; R) = \sum_{\psi \in \mathcal{B}} \left|\langle |\phi_j|^2, \psi\rangle\right|^2 |h_R(t_\psi)|^2 + \frac{1}{12}\int_{-\infty}^{\infty} \left|\left\langle |\phi_j|^2, E\left(\cdot, \frac{1}{2} + it\right)\right\rangle\right|^2 |h_R(t)|^2 \, dt.$$

Need to show that if $R \gg t_j^{-\delta}$ with $\delta < 1$, both terms are $o(1)$ as $t_j \to \infty$.

Spectral sum only involves nonnegative terms, so no unexploited additional cancellation to stop us from getting to the Planck scale.

Small Scale QUE Almost Everywhere

Asymptotics for $h_R(t)$: small if $t \gg R^{-1}$, so we only need to worry about $\psi \in \mathcal{B}$ with $t_\psi \ll R^{-1}$.

For $|\langle |\phi_j|^2, \psi \rangle|^2$, Watson–Ichino gives

$$|\langle |\phi_j|^2, \psi \rangle|^2 = |\mathcal{A}_{\phi_j,\psi}|^2 |\mathcal{S}_{\phi_j,\psi}|^2.$$

Spectral part satisfies

$$|\mathcal{S}_{\phi_j,\psi}|^2 \approx \frac{1}{(1+t_\psi)(|4t_j^2 - t_\psi^2| + 1)^{1/2}} \times \begin{cases} 1 & \text{if } 0 < t_\psi < 2t_j, \\ e^{-\pi(t_\psi - 2t_j)} & \text{if } t_\psi \geq 2t_j. \end{cases}$$

Assuming GLH, arithmetic part satisfies

$$|\mathcal{A}_{\phi_j,\psi}|^2 \ll_\varepsilon (|4t_j^2 - t_\psi^2| + 1)^\varepsilon.$$

So only range in which this is not so small as to be innocuous is $t_\psi \asymp 2t_j$: constructive interference.
Does not occur if $R \gg t_j^{-\delta}$ with $\delta < 1$!

Small Scale QUE Almost Everywhere

Upshot: GLH implies that for R bigger than the Planck scale,

$$\mathrm{Var}(\phi_j; R) = \sum_{\psi \in \mathcal{B}} \left| \langle |\phi_j|^2, \psi \rangle \right|^2 |h_R(t_\psi)|^2 + \frac{1}{12} \int_{-\infty}^{\infty} \left| \left\langle |\phi_j|^2, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle \right|^2 |h_R(t)|^2 \, dt$$

involves terms that are small, and so $\mathrm{Var}(\phi_j; R) = o(1)$.

Spectral Expansion of the Fourth Moment

For the fourth moment, Parseval gives

$$\int_{\Gamma\backslash\mathbb{H}} |\phi_j(z)|^4 \, d\mu(z) = 1 + \sum_{\psi \in \mathcal{B}} \left| \langle |\phi_j|^2, \psi \rangle \right|^2$$
$$+ \frac{1}{12} \int_{-\infty}^{\infty} \left| \left\langle |\phi_j|^2, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle \right|^2 dt.$$

Same as $\mathrm{Var}(\phi_j; R)$, except additional term 1 and no $|h_R(t)|^2$.

Need to show latter two terms are $2 + o(1)$ as $t_j \to \infty$.
By the same argument as for the variance, the only range that is not small is $t_\psi \asymp 2t_j$. Watson–Ichino, Stirling, GLH, and Weyl law show that

$$\int_{\Gamma\backslash\mathbb{H}} |\phi_j(z)|^4 \, d\mu(z) \ll_\varepsilon t_j^\varepsilon.$$

More work to treat the range $t_\psi \asymp 2t_j$ yields the desired asymptotic; uses heavily arithmetic nature of $\mathcal{A}_{\phi_j,\psi}$.

Unconditional Results

Required GLH in order to satisfactorily bound $\mathcal{A}_{\phi_j,\psi}$. Can we say anything unconditionally?
Divide up spectral sum into dyadic ranges!

Quantification of rate of equidistribution for QUE: the rate $R = t_j^{-\delta}$ for small scale QUE almost everywhere is equivalent to *average* bounds

$$\sum_{\substack{\psi \in \mathcal{B} \\ T \leq t_\psi \leq 2T}} |\mathcal{A}_{\phi_j,\psi}|^2 \ll T t_j^{1-\delta'}$$

uniformly in $T \ll R^{-1}$ (any $\delta' > 0$ suffices!). The larger the value of T, the better the rate.

Best known *unconditional* bound for $|\mathcal{A}_{\phi_j,\psi}|^2$ is the convexity bound:

$$|\mathcal{A}_{\phi_j,\psi}|^2 \ll (t_\psi + 1)(|4t_j^2 - t_\psi^2| + 1)^{1/2}.$$

Gives no nontrivial information.

Unconditional Results

For $\phi_j(z) = E(z, 1/2 + it_j)$ an Eisenstein series or ϕ_j a dihedral Maaß form associated to a Größencharakter of a real quadratic field $\mathbb{Q}(\sqrt{D})$, such average bounds can be proven.

Reason: the degree 6 L-function $L(1/2, \mathrm{sym}^2\phi_j \otimes \psi)$ factorises as

- the product of three degree 2 L-functions if ϕ_j is an Eisenstein series,
- the product of a degree 2 and a degree 4 L-function if ϕ_j is dihedral.

These L-functions are much less complicated.

Unconditional Results

Reason for success in bounds for sums of $|\mathcal{A}_{\phi_j,\psi}|^2$ when ϕ_j is Eisenstein or dihedral: many ways to apply Hölder's inequality due to factorisation of $L(1/2, \mathrm{sym}^2\phi_j \otimes \psi)$.

Proof uses deep pre-existing results on subconvexity (Michel–Venkatesh), plus many arithmetic tools:

- Kuznetsov trace formula,
- approximate functional equation,
- Voronoĭ summation formula,
- spectral decomposition of sums of Kloosterman sums,
- spectral large sieve. . .

Nonarithmetic Surfaces

Results depend heavily on tools from analytic number theory; only applicable when $M = \Gamma\backslash\mathbb{H}$ is an arithmetic surface.

These results look hopelessly difficult to prove in nonarithmetic settings.

Question

Can one show the failure of QUE at scales below the Planck scale for nonarithmetic surfaces?

Thank you!