Min-max harmonic maps and extremal metrics for Laplacian eigenvalues

(joint with Mikhail Karpukhin)

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Isoperimetric problems for Laplacian eigenvalues

On a closed (2-dimensional) surface (M^2, g), consider the Laplacian

$$\Delta_g = d^* d$$

with positive spectrum

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \cdots .$$

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'Isoperimetric' Problem:

- *Find upper bounds for $\bar{\lambda}_k$ over all metrics of fixed conformal type, or over all metrics on a surface M of fixed topology.*
- *Identify/characterize maximizing metrics, or maximizing sequences saturating these bounds.*

Isoperimetric problems for Laplacian eigenvalues

Given M^2 and conformal class $[g]$, define the *conformal eigenvalues*

$$\Lambda_k(M,[g]) := \sup_{g\in[g]} \bar{\lambda}_k(M,g)$$

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Study of these maximization problems on closed surfaces begins with the classical result of Hersch:

Theorem (Hersch '70)

The round metric maximizes $\bar{\lambda}_1(S^2,g)$ among all metrics on S^2. I.e., $\Lambda_1(S^2)=8\pi$.

Proof Idea: For any conformal diffeomorphism

$$\Phi : (S^2, g) \to S^2_{std} \subset \mathbb{R}^3,$$

can compose with $F \in Conf(S^2_{std})$ such that the components Ψ^i of the composition $\Psi = F \circ \Phi$ satisfy $\int_{S^2} \Psi^i dv_g = 0$.

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By the variational characterization

$$\lambda_1(M, g) := \inf\{\frac{\|d\varphi\|^2_{L^2(M,g)}}{\|\varphi\|^2_{L^2(M,g)}} \mid \int_M \varphi dv_g = 0\}$$

of $\lambda_1(M, g)$, we then have

$$\int |d\Psi^i|^2 \geq \lambda_1(M, g) \int_{S^2} |\Psi^i|^2 dv_g,$$

and summing over $i = 1, 2, 3$ gives

$$2 \cdot Area(S^2_{std}) = \int |d\Psi|^2 \geq \lambda_1(M, g) Area(S^2, g). \ \Box$$

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Namely, given a branched conformal immersion $\phi : (M^2, g) \to S^n$, one can always find $F \in Conf(S^n)$ for which

$$\int_M (F \circ \phi)^1 dv_g = \cdots = \int_M (F \circ \phi)^{n+1} dv_g = 0,$$

so that

$$Area(F \circ \phi(M)) = E(F \circ \phi) \geq \frac{1}{2}\lambda_1(M, g)Area(M, g).$$

Li–Yau define the nth *conformal volume* $V_c(n, M, [g])$ of a conformal class $[g]$ on a surface M by

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Theorem (P. Li–S.T. Yau '82)

For any conformal structure $(M, [g])$, and any $g \in [g]$,

$$2V_c(n, M, [g]) \geq \bar{\lambda}_1(M, g),$$

with equality if and only if g is induced by a minimal immersion $M \to S^n$ by first eigenfunctions.

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- $V_c(n, M, [g])$ also gives a lower bound for the Willmore energy of conformal immersions $M \to S^n$, linking maximization of $\bar{\lambda}_1$ and minimization of $\mathcal{W}$.

In the years since, a large literature has grown up around these problems; let's quickly highlight a few key contributions. *(For a nice overview, check out the slides or video from Iosif Polterovich's May 25 talk in this seminar.)*

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- (Korevaar '93): There is a universal constant C such that

$$\bar{\lambda}_k(M, g) \leq C(\gamma + 1)k$$

  for any metric g on a closed, orientable surface M of genus γ. In particular, $\Lambda_k(M) < \infty$ for every k.

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The latter result was the first to rely on *existence theory* for maximizing metrics–a subject which has exploded over the last decade.

Existence theory for conformally maximal metrics

By work of Petrides and (via different techniques) Nadirashvili-Sire and Karpukhin-Nadirashvili-Penskoi-Polterovich, the existence theory for metrics achieving $\Lambda_k(M, [g])$ is fairly well-understood: in general,

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- For $k = 1$ and $M \neq S^2$, strict inequality ($\Lambda_1 > 8\pi$) holds; hence, every conformal class on a closed Riemannian surface admits a $\bar{\lambda}_1$-maximizing metric.

Existence theory for conformal maximizers also plays a key role in the existence theory for *topological* maximizers–as in Mathiessen-Siffert's work establishing the existence of globally maximizing metrics for $\bar{\lambda}_1(M,g)$.

Existence theory for conformally maximal metrics

In particular, for every conformal structure (M, c), maximization of $\bar{\lambda}_k$ gives rise (modulo bubbling phenomena for $k \geq 2$) to a harmonic sphere-valued map $\Phi : M \to S^{N(c)}$ of energy

$$E(\Phi) := \frac{1}{2}\int_M |d\Phi|^2 dv_g = \frac{1}{2}\Lambda_k(M, c),$$

from which one recovers the associated maximal metric $\bar{g}$ by setting

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Recall that a sphere-valued map $\Phi : (M, g) \to S^n$ is *harmonic* if (and only if) it is a critical point of the Dirichlet energy

$$E(\Phi) := \frac{1}{2}\int_M |d\Phi|^2,$$

viewed as a functional on S^N-valued maps; equivalently,

$$\Delta_g \Phi = |d\Phi|_g^2 \Phi \text{ as maps } M \to \mathbb{R}^{n+1}.$$

As critical points of the Dirichlet energy, harmonic maps arise naturally from *variational methods*–i.e., minimization, or (more generally) Morse-theoretic/min-max methods–applied to the energy functional an appropriate space of maps.

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- For the first two eigenvalues, yes: $\Lambda_1(M, c)$ and $\Lambda_2(M, c)$ can be identified with natural min-max energies associated to certain families of sphere-valued maps.
- Using the min-max characterization, we can show that extremal metrics satisfy a stronger maximization property, allowing us to compare the quantities $\Lambda_k(M, c)$ to the spectra of some other natural pseudodifferential operators.

Min-max characterization of $\Lambda_1(M, c)$.

For $n \geq 2$, consider a *weakly continuous* family of maps

$$\bar{B}^{n+1} \ni a \mapsto F_a \in W^{1,2}(M, S^n)$$

satisfying

$$F_a \equiv a \text{ for } a \in S^n.$$

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Example: One could take $F_a = G_a \circ \phi$, where $\phi : M \to S^n$ is a branched conformal immersion and $G_a(x) = a + \frac{(1-|a|^2)}{|x+a|^2}(x + a)$. Note that the maps G_a generate the conformal automorphisms of S^n modulo isometries.

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The map

$$\bar{B}^{n+1} \ni a \mapsto \frac{1}{Area(M)} \int_M F_a dv_g \in \mathbb{R}^{n+1}$$

is continuous, and restricts to the identity $S^n \to S^n$, so there must be $a \in B^{n+1}$ for which $\int_M F_a = 0 \in \mathbb{R}^{n+1}$.

In particular, for any such family $\overline{B}^{n+1} \ni a \mapsto F_a$, we see that

$$\lambda_1(M,g) Area(M,g) = \lambda_1(M,g) \int_M |F_a|^2 \leq \int_M |dF_a|^2$$

at the point $a \in \overline{B}^{n+1}$ where $\int_M F_a = 0 \in \mathbb{R}^{n+1}$.

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Hence, defining a min-max energy

$$\mathcal{E}_n(M,g) := \inf_F \sup_{a \in B^{n+1}} E(F),$$

over the collection of all such families, it follows that

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Noting that $\mathcal{E}_n(M,g)$ is conformally invariant, we see that

$$\frac{1}{2}\Lambda_1(M,[g]) \leq \mathcal{E}_n(M,[g]) \leq V_c(n,M,[g]).$$

In practice, we opt for a slightly different definition of $\mathcal{E}_n(M,[g])$, which we can identify as the energy of a harmonic map via min-max methods...

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Modulo technicalities, if all three are satisfied, then the quantity

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$$c := \inf_{A \in \Gamma} \max_{a \in A} f(a)$$

is achieved as $f(x_\Gamma)$ for a critical point $x_\Gamma \in Crit(f)$. (Intuitively, otherwise we could use the gradient flow to find a family $A \in \Gamma$ with $\max_{a \in A} f(a) < c$.)

Since the space $W^{1,2}(M, S^n)$ with the weak topology is far from a Banach/Hilbert manifold (and the Dirichlet energy is not a smooth, Palais-Smale functional on this space), we define our min-max energies by a regularization procedure.

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Given $\epsilon > 0$ and $F : M \to \mathbb{R}^{n+1}$, consider the *Ginzburg-Landau* energy

$$E_\epsilon(F) := \int_M \frac{1}{2}|dF|^2 + \frac{(1-|F|^2)^2}{4\epsilon^2}.$$

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For each $\epsilon > 0$, $\mathcal{E}_{n,\epsilon}(M^2, g)$ is achieved by a critical point $\Phi_\epsilon \in C^\infty(M, \mathbb{R}^{n+1})$ for E_ϵ of Morse index $\leq n+1$. For $n \geq 2$, taking $\epsilon \to 0$ (appealing to results of Lin–Wang), we find harmonic maps $\Phi : M \to S^n$ and $\phi_1, \ldots, \phi_k : S^2 \to S^n$ satisfying

$$E(\Phi) + \Sigma_{i=1}^k E(\phi_i) = \mathcal{E}_n(M,[g])$$

and the Morse index bound

$$ind_E(\Phi) + \Sigma_{i=1}^k ind_E(\phi_i) \leq n+1.$$

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Aside: In the *scalar-valued* ($n = 0$) case, the min-max energies $\mathcal{E}_{n,\epsilon}(M, g)$ were introduced by Marco Guaraco in his thesis work. In this case, the energies blow up like $\frac{1}{\epsilon}$ as $\epsilon \to 0$, and critical points concentrate along minimal hypersurfaces in M (geodesic networks, if M is a surface).

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In the complex-valued case ($n = 1$), the energies $\mathcal{E}_{n,\epsilon}$ have been studied by Da Rong Cheng and myself. In this case, $\mathcal{E}_{n,\epsilon}(M, g)$ blows up like $\log(1/\epsilon)$ as $\epsilon \to 0$, and critical points exhibit energy concentration along (weak) minimal submanifolds of codimension two in M (or a collection of points critical for a certain interaction energy, if $\dim(M) = 2$).

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In particular, since our min-max harmonic map $\Phi : M \to S^n$ and bubble maps $\phi_1, \ldots, \phi_k : S^2 \to S^n$ satisfy

$$E(\Phi) + \Sigma_{i=1}^k E(\phi_i) \geq \frac{1}{2}\Lambda_1(M, [g])$$

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we can rule out the presence of bubbles for n sufficiently large...

Theorem (Karpukhin-S. '20)

For every conformal structure $(M,[g])$ *on a closed surface, if* $n > 5$*, there exists a harmonic map* $\Phi_n : M \to S^n$ *of energy*

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The key claim now is that $\mathcal{E}_n$ *coincides* with $\frac{1}{2}\Lambda_1$ for n large. To this end, we first show that the maps Φ_n stabilize in an appropriate sense as $n \to \infty$.

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Proposition

For any conformal class $[g]$ on M and any $K<\infty$, there exists $N(M,[g],K) \in \mathbb{N}$ such that every sphere-valued harmonic map $\Phi : M \to S^n$ of energy $E(\Phi) \leq K$ factors through a totally geodesic embedding $M \to S^N \hookrightarrow S^n$.

For the harmonic maps $\Phi_n : M \to S^n$ constructed above, consider the associated metrics (possibly with conical singularities)

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The smallest integer k such that $\lambda_k(M, \bar{g}_n) = 2$ is called the *spectral index* $ind_S(\Phi_n)$. Equivalently, for any metric $g \in [g]$, it is the index of the Schrödinger operator $\Delta_g - |d\Phi|_g^2$.

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For $k = ind_S(\Phi_n)$, note that

$$\bar{\lambda}_k(M, \bar{g}_n) = 2\mathcal{E}_n(M, [g]) \geq \Lambda_1(M, [g]).$$

If we can show that $k = 1$, then $\bar{\lambda}_1(M, \bar{g}_n) \leq \Lambda_1(M, [g])$ as well, and we arrive at the desired equality

$$\mathcal{E}_n = 2\Lambda_1.$$

To show that $ind_S(\Phi_n) = 1$ for n large, we note that $\mathcal{E}_n(M, [g])$ is decreasing in n, so that $\mathcal{E}_n(M, [g]) \leq K$ for all $n \geq 2$.

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By the stabilization lemma, there exists $N = N(M, [g])$ such that, for $n > N$, the maps $\Phi_n : M \to S^n$ factor

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Comparing the Morse indices $ind_E(\Phi_n)$ and $ind_E(\Psi_n)$ as critical points of the energy functional, one finds

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In particular, it follows that

$$(n+1) \geq ind_E(\Phi_n) \geq (n - N)ind_S(\Phi_n),$$

and consequently $ind_S(\Phi_n) = 1$ for $n > 2N + 1$.

Theorem (Karpukhin-S. '20)

For n sufficiently large, $\mathcal{E}_n(M,[g]) = \frac{1}{2}\Lambda_1(M,[g])$*, and the metric* $g_{\Phi_n} = \frac{1}{2}|d\Phi_n|_g^2 g$ *associated to the harmonic map* $\Phi_n : M \to S^n$ *maximizes* $\bar{\lambda}_1(M,g)$ *in* $[g]$*.*

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Rough proof idea: If not, we could find infinitely many $n \in \mathbb{N}$ and harmonic maps $\Phi_n : M \to S^n$ with $E(\Phi_n) \leq K$, for which the kernel of $\Delta_g - |d\Phi_n|_g^2$ has dimension $\geq n+1$.

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By a variant of the usual bubbling analysis for harmonic maps, we deduce the existence of a function $\rho \in L^p(M \cup S^2 \cup \cdots \cup S^2)$ $(p > 1)$ such that the Schrödinger operator $\Delta - \rho$ has infinitely many eigenvalues ≤ 0. This cannot occur. $\square$

Applications of the min-max characterization

By definition, $\Lambda_1(M, [g])$ is the supremum of $\bar{\lambda}_1(M, g)$ among all $g \in [g]$. Using the min-max characterization, we can identify $\Lambda_1(M, [g])$ as the supremum of a much larger class of "first eigenvalues" generalizing $\bar{\lambda}_1(M, g)$.

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Following G. Kokarev, given a conformal class (M,c) and a Radon probability measure $\mu \in [C^0(M)]^*$, one may define a "first eigenvalue"

$$\lambda_1(M,c,\mu) := \inf\{\frac{\int_M |d\varphi|^2 dv_g}{\|\varphi\|^2_{L^2(\mu)}} \mid \int_M \varphi d\mu = 0\}, \tag{1}$$

where $g \in c$ is arbitrary.

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where $g \in c$ is arbitrary.

If $\mu = \rho^2 dv_g$ for $0 < \rho \in C^\infty(M)$, this is just the first eigenvalue $\lambda_1(M,\rho^2 g)$, but the definition includes some other quantities of geometric interest.

Example: If (M, g) is a surface with boundary (normalized to have unit length) and $\mu = \mathcal{H}^1|_{\partial M}$, then $\lambda_1(M, c, \mu) = \sigma_1(M, c)$ is precisely the *first Steklov eigenvalue* of (M, g)–that is, the first eigenvalue of the Dirichlet-to-Neumann map

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Among other results, he observed that the Li–Yau conformal volume bound extends to arbitrary probability measures, so that

$$\lambda_1(M, c, \mu) \leq 2V_c(M, c).$$

Definition

Call a Radon measure μ *admissible* if the map $C^1(M) \to L^2(\mu)$ extends to a compact map $W^{1,2}(M) \to L^2(\mu)$.

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Theorem (Karpukhin-S. '20)

For any admissible probability measure μ, $\lambda_1(M, c, \mu) \leq 2\mathcal{E}_n(M, c)$, *with equality only if* $\mu = \frac{1}{2E(\Phi)}|d\Phi|_g^2 dv_g$ *for a harmonic map* $\Phi : M \to S^n$ *of spectral index one. In particular,*

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In particular, we see that maximizers of $\lambda_1(M, c, \mu)$ coincide with maximizers of $\bar{\lambda}_1(M, g)$, establishing regularity of arbitrary maximizing measures.

Steklov applications

For any domain $\Omega \subset (M, g)$, it's easy to see that

$$|\partial\Omega|\sigma_1(\Omega, g) \leq \lambda_1(M, [g], \frac{\mathcal{H}^1 \lfloor \partial\Omega}{|\partial\Omega|}),$$

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Letting

$$\Sigma_1(M_{\gamma,k}) := \sup_{g \in Met(M_{\gamma,k})} Length(\partial M_{\gamma,k}, g)\sigma_1(M_{\gamma,k}, g),$$

it follows that

$$\Sigma_1(M_{\gamma,k}) \leq \Lambda_1(M_\gamma).$$

At the same time, Girouard and Lagacé showed that the *opposite* bound holds as $k \to \infty$. (Indeed, they show that any closed surface (M, g) contains a sequence of subdomains Ω_k for which $|\partial\Omega_k|\sigma_1(\Omega_k, g) \to Area(M)\lambda_1(M, g)$.)

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Corollary

$$\lim_{k\to\infty} \Sigma_1(M_{\gamma,k}) = \sup_{k\in\mathbb{N}} \Sigma_1(M_{\gamma,k}) = \Lambda_1(M_\gamma). \qquad (2)$$

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The study of maximal metrics for σ_1 has received significant attention following the work of Fraser-Schoen, who observed that maximizing metrics are induced by free-boundary minimal immersions $M_{\gamma,k} \to B^n$ into Euclidean balls.

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Equation (2) tells us that, for fixed genus γ, the areas of the associated f.b.m.s's are bounded above by the area of the minimal surface of genus γ in S^n realizing $\Lambda_1(M_\gamma)$, with inequality approaching equality as #(boundary components)$\to \infty$.

Second eigenvalues

Taking intuition from arguments used by Nadirashvili and Petrides to characterize $\Lambda_2(S^2)$, we define a collection $\Gamma_{n,2}(M)$ of weakly continuous families

$$[\overline{B}^{n+1}]^2 \ni (a,b) \mapsto F_{a,b} \in W^{1,2}(M,S^n)$$

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where $\tau_b \in O(n+1)$ denotes reflection through the hyperplane perpendicular to b. Roughly, we define

$$\mathcal{E}_{n,2}(M, [g]) := \inf_{F \in \Gamma_{n,2}(M)} \sup_{(a,b)} E(F_{a,b}).$$

To see that $\mathcal{E}_{n,2}(M,[g]) \geq \frac{1}{2}\Lambda_2(M,[g])$, for any metric $g \in [g]$ with first eigenfunction ϕ_1, we wish to show that the map

$$\mathcal{I} : [\overline{B}^{n+1}]^2 \to \mathbb{R}^{2(n+1)},\ \mathcal{I}(a,b) := -\left(\int_M F_{a,b} dv_g, \int_M \phi_1 F_{a,b}\right)$$

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has a zero for every $F \in \Gamma_{n,2}(M)$. Indeed, if $\mathcal{I}(a,b) = 0$, then the estimate

$$2E(F_{a,b}) \geq \lambda_2(M,g) \int_M |F_{a,b}|^2 = \lambda_2(M,g) Area(M,g)$$

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Suppose, to the contrary, that $\mathcal{I}$ has no zeroes. Then, identifying $[B^{n+1}]^2$ and B^{2n+2} via

$$\beta : [B^{n+1}]^2 \to B^{2n+2},\ \beta(a,b) := \frac{|(a,b)|}{\max\{|a|,|b|\}}(a,b),$$

we can define a map $\mathcal{J} : \overline{B}^{2n+2} \to S^{2n+1}$ by

$$\mathcal{J}(a,b) = \frac{\mathcal{I}(\beta^{-1}(a,b))}{|\mathcal{I}(\beta^{-1}(a,b))|}.$$

In particular, the restriction $\mathcal{J}|_S : S^{2n+1} \to S^{2n+1}$ to the boundary defines a null-homotopic self-map of S^{2n+1}.

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Thus, (modulo some technical work of ensuring transversality conditions), we reach a contradiction, and conclude that the original map $\mathcal{I}$ must have a zero. In particular,

$$\mathcal{E}_{n,2}(M,[g]) \geq \frac{1}{2}\Lambda_2(M,[g]).$$

Of course, we also need to show that families $F \in \Gamma_{n,2}(M)$ satisfying these conditions *exist*.

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Following ideas of Nadirashvili, one can construct explicit families of maps in $\Gamma_{n,2}(M)$ via a composition

$$F_{a,b} = G_a \circ \Psi_b \circ \phi,$$

where

- $\phi : M \to S^n$ is a branched conformal immersion,
- G_a is the familiar $(n+1)$-parameter family of conformal dilations,
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In this way, one finds that $\Gamma_{n,2}(M) \neq \varnothing$, and moreover,

$$\frac{1}{2}\Lambda_2(M,[g]) \leq \mathcal{E}_{n,2}(M,[g]) \leq 4V_c(n,M,[g]).$$

Applying the same machinery from before, we show that

Theorem (Karpukhin-S. '20)

For any closed Riemann surface $(M,[g])$ *and* $n \geq 9$*, there exists a harmonic map* $\Psi_n : M \to S^n$ *such that either*

$$\frac{1}{2}\Lambda_2(M,[g]) \leq \mathcal{E}_{n,2}(M,[g]) = E(\Psi_n) \text{ and } ind_E(\Psi_n) \leq 2n+2$$

or

$$\frac{1}{2}\Lambda_2(M,[g]) \leq \mathcal{E}_{n,2}(M,[g]) = E(\psi_n) + 4\pi \text{ and } ind_E(\Psi_n) \leq n+4.$$

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For $n \geq N(M,[g])$ *sufficiently large,*

$$\mathcal{E}_{n,2}(M,[g]) = \frac{1}{2}\Lambda_2(M,[g]).$$

We can then argue as we did for the first eigenvalue to see that

- $\Lambda_2(M, [g])$ is an upper bound for $\lambda_2(M, [g], \mu)$ for all admissible probability measures μ. (With equality only for the energy densities of harmonic maps of spectral index 2.)

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We expect that similar min-max characterizations can be obtained for the higher eigenvalues $\Lambda_k(M,[g])$. If this is correct, then all of the applications–Steklov bounds, characterization of λ_k-maximal measures–should extend to the full spectrum.

Thank you!