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Inverse Steklov spectral problem on curvilinear polygons

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Spectral Geometry in the Clouds archimede.mat.ulaval.ca/agirouard/SpectralClouds

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References

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The Steklov spectrum is discrete, $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow +\infty$, the corresponding normalised quadratic form is $\dfrac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\partial\Omega)}}$.

D–N map

Alternatively, Steklov eigenvalues can be viewed as the eigenvalues of the Dirichlet-to-Neumann map,

$$\mathcal{D} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$
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Example (disk)

Steklov spectrum of a disk of radius one is $0, 1, 1, 2, 2, 3, 3, \dots, m, m, \dots$, eigenfunctions being $1, r\sin\phi, r\cos\phi, \dots, r^m \sin m\phi, r^m \cos m\phi, \dots$. For radius R, scale as $1/R$. Eigenfunctions decay fast in the interior.

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(Girouard & Polterovich 2017). (Non-trivial) separation of variables.

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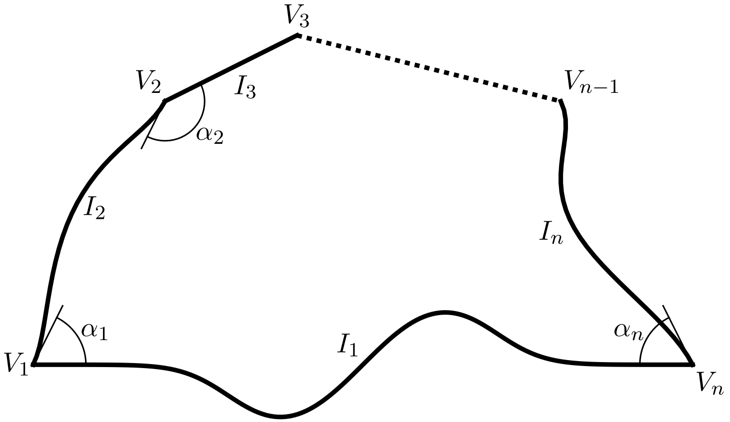

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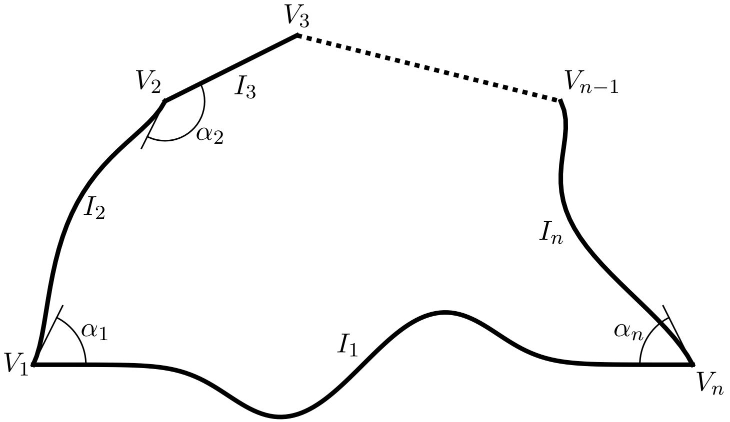

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0, \pi)^n, \boldsymbol{\ell} = (\ell_1, \ldots, \ell_n) \in \mathbb{R}_+^n, \ell_j = |I_j|, L = \sum_{j=1}^{n} \ell_j$$

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Exceptional and special angles

Although they do not appear explicitly above, particular roles in our analysis are played by exceptional angles

$$\alpha \in \mathcal{E} := \left\{ \frac{\pi}{2k}, k \in \mathbb{N} \right\}$$

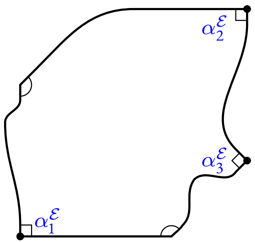
and special angles

$$\alpha \in \mathcal{S} := \left\{ \frac{\pi}{2k+1}, k \in \mathbb{N} \right\}$$

Both exceptional and special angles can be even or odd depending on whether the corresponding k is even or odd.

Parity of $\alpha \in \mathcal{E} \cup \mathcal{S}$ is $\mathcal{O}(\alpha) := (-1)^k$.

Exceptional case: needs re-labelling



- Exceptional angles $\alpha_1^{\mathcal{E}}, \ldots \alpha_K^{\mathcal{E}}$

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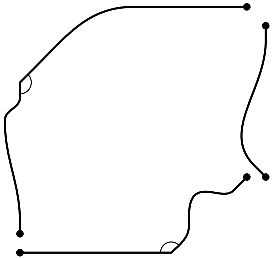

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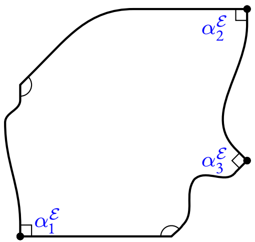

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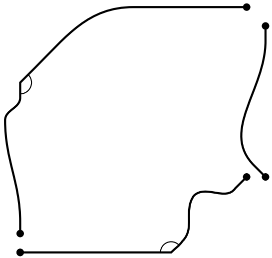

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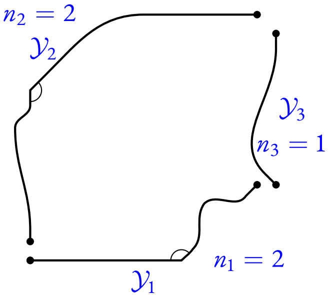

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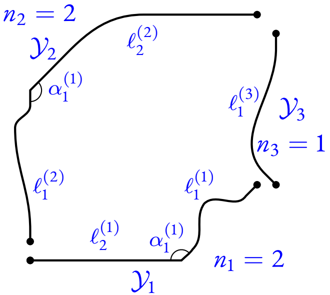

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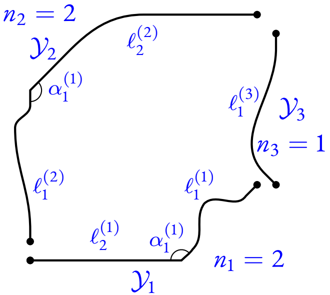

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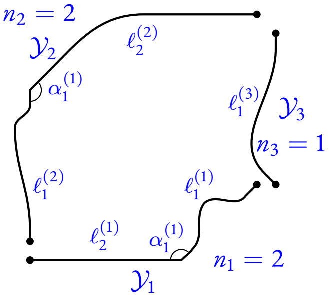

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- Assume we have K_{even} even exceptional boundary components and K_{odd} odd exceptional boundary components, $K_{\text{even}} + K_{\text{odd}} = K$.

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Eigenfunc.	(in a sense) equidistributed over the whole boundary	concentrate on some exceptional boundary components

Direct problem summary

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curvilinear polygon $\mathcal{P}$

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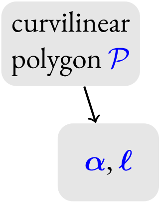

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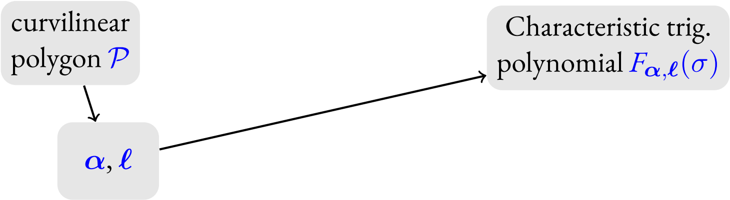

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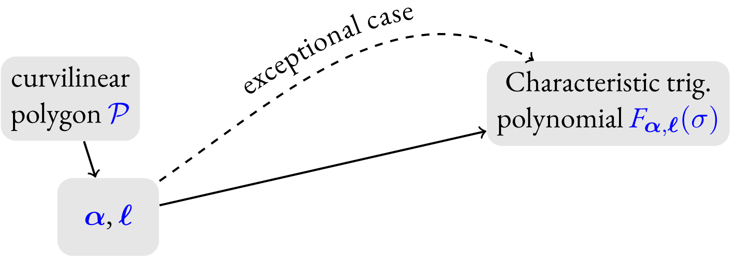

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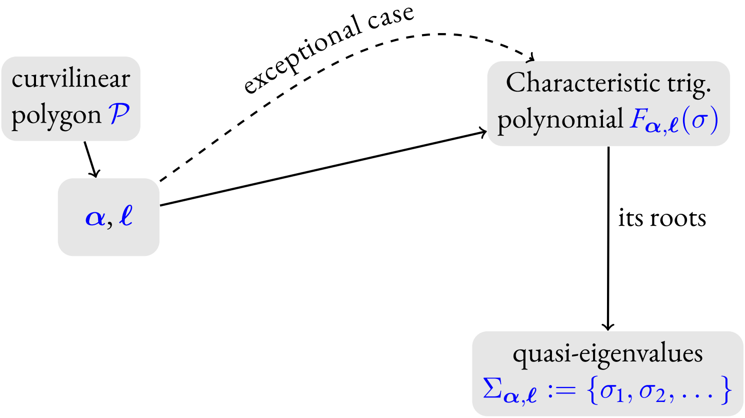

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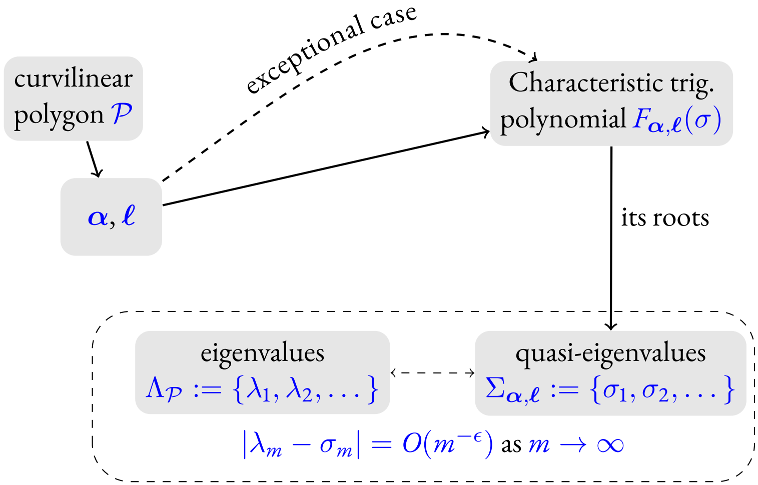
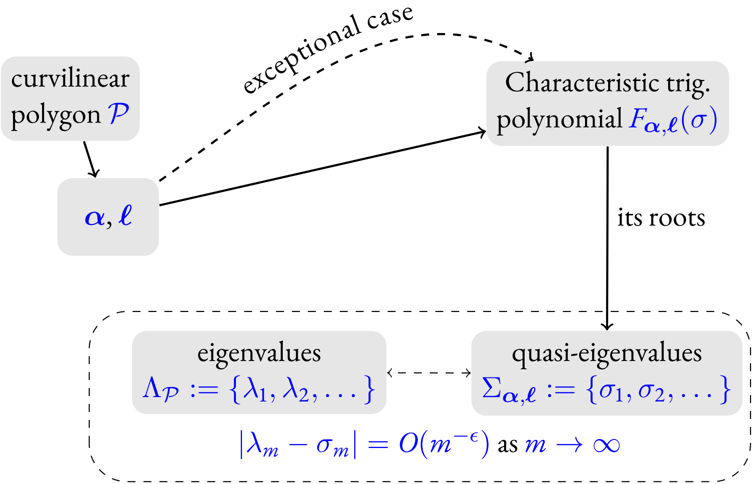


Direct problem summary and inverse problems statements



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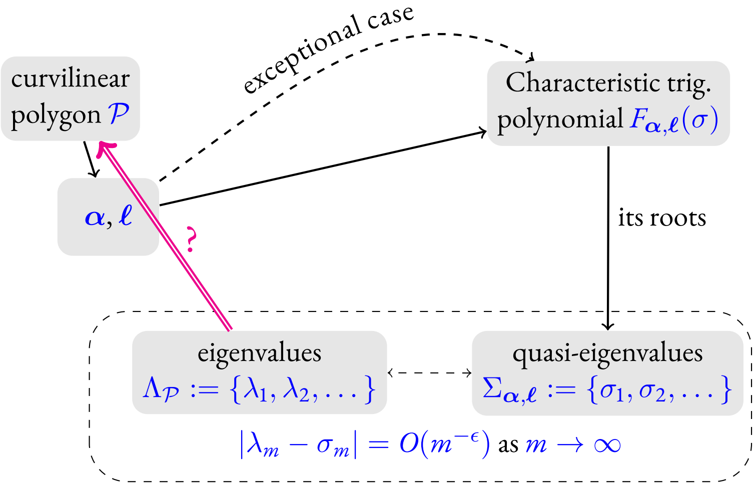

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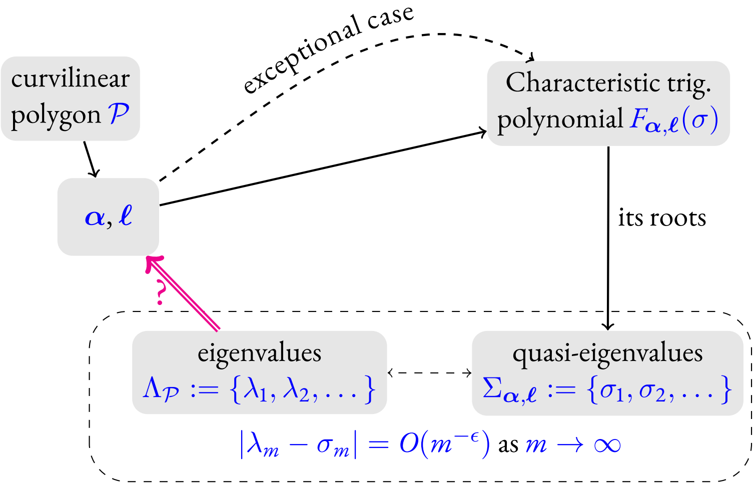

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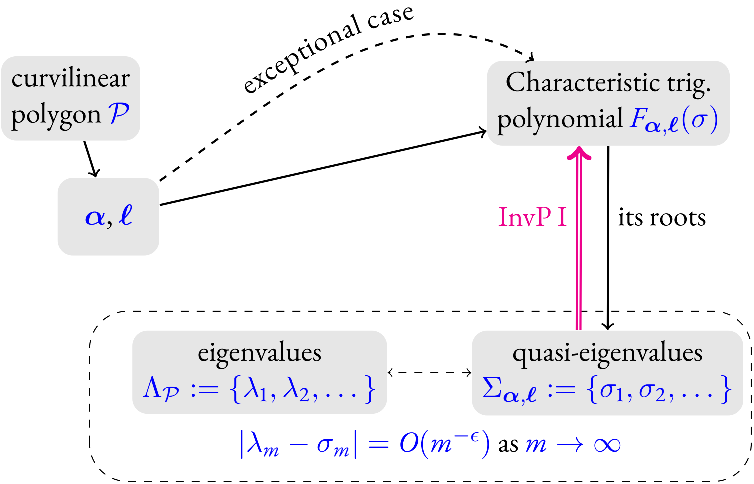

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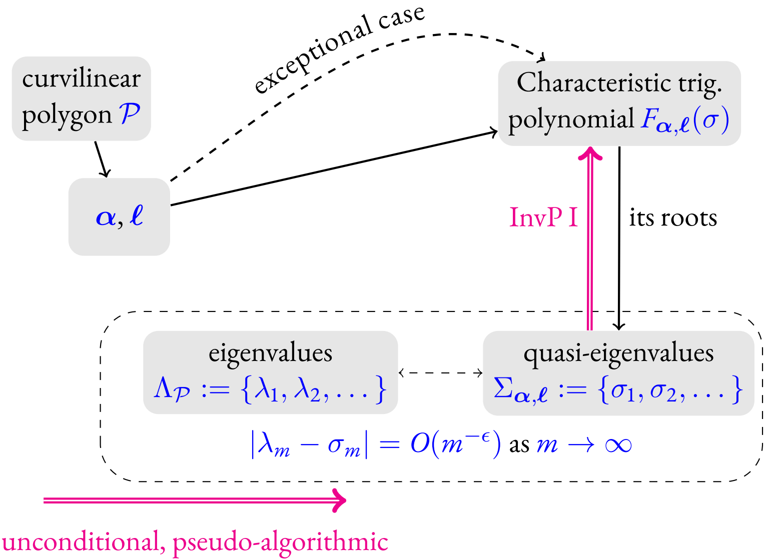

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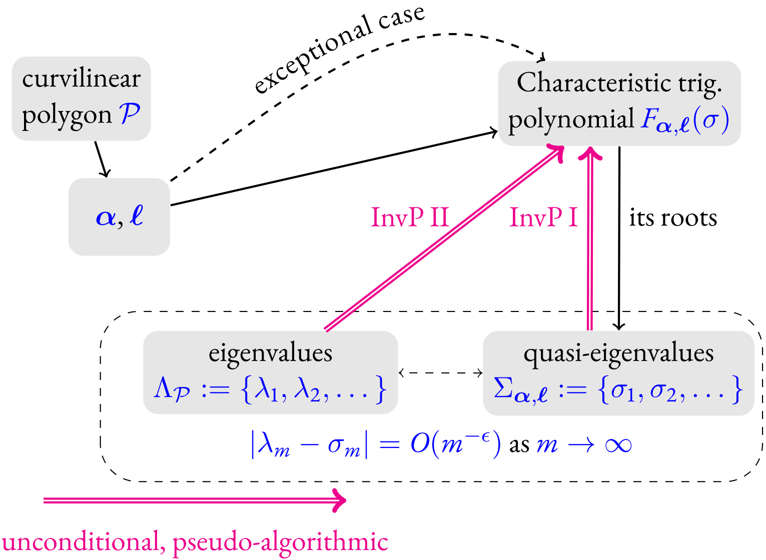

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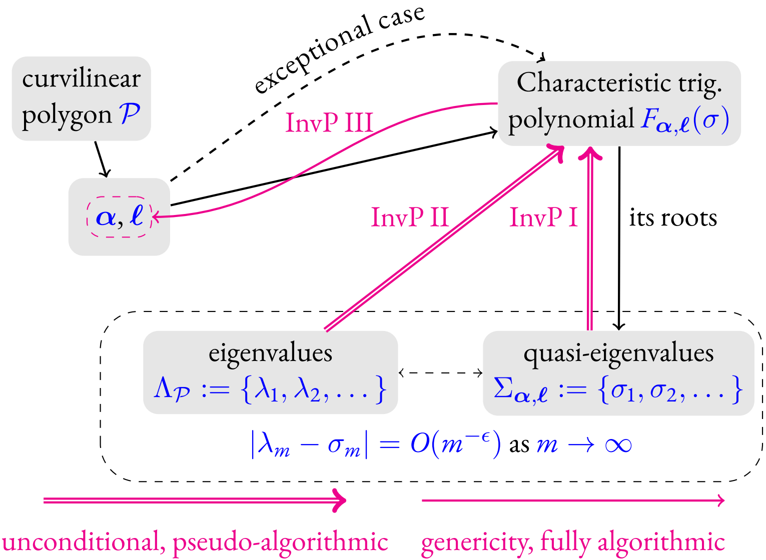

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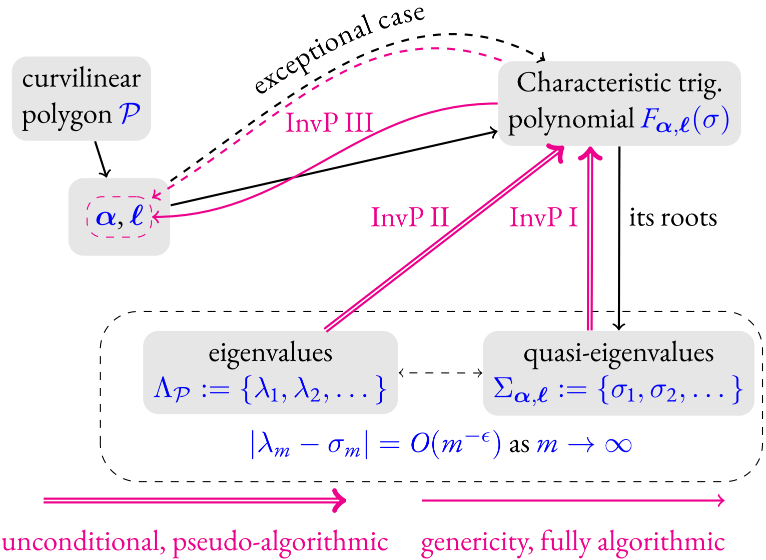

Inverse problems — isospectrality

Definition (Steklov isospectrality and and quasi-isospectrality)

We say that two domains Ω_1 and Ω_2 are (Steklov) isospectral if their Steklov spectra coincide, $\Lambda_{\Omega_1} = \Lambda_{\Omega_2}$.

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Definition (Loose equivalence)

We say that two curvilinear polygons $\mathcal{P}(\boldsymbol{\alpha}, \boldsymbol{\ell})$ and $\tilde{\mathcal{P}}(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\ell}})$ are loosely equivalent if one can choose the orientation and the enumeration of vertices of these polygons in such a way that $\boldsymbol{\ell} = \tilde{\boldsymbol{\ell}}$ and either $\mathbf{c}_{\boldsymbol{\alpha}} = \mathbf{c}_{\tilde{\boldsymbol{\alpha}}}$ or $\mathbf{c}_{\boldsymbol{\alpha}} = -\mathbf{c}_{\tilde{\boldsymbol{\alpha}}}$.

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We will assume, at some stage, that our polygons satisfy two generic conditions:

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Subject to admissibility conditions, we have …

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$\boldsymbol{\ell}$ and $\tilde{\boldsymbol{\ell}}$ are the same modulo a cyclic shift and a change of orientation; cosine vectors $\mathbf{c}_{\boldsymbol{\alpha}}$ and $\mathbf{c}_{\tilde{\boldsymbol{\alpha}}}$ are the same up to a change of sign

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Corollary

Given the spectrum Λ of an admissible non-exceptional polygon $\mathcal{P}$, we can recover its number of vertices, side lengths up to change of orientation and cyclic shifts, and the cosine vector up to a change of sign.

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Let $\mathcal{P}$ and $\tilde{\mathcal{P}}$ be two quasi-isospectral admissible curvilinear polygons. Assume additionally that $\mathcal{P}$ is not exceptional. Then $\mathcal{P}$ and … nt.

This means we have formulae/algorithms for that but some steps may be non-trivial to realise numerically

Corollary

Given the spectrum Λ of an admissible non-exceptional polygon $\mathcal{P}$, we can recover its number of vertices, side lengths up to change of orientation and cyclic shifts, and the cosine vector up to a change of sign.

Main Theorem, simplified variant

Before the main result, I state

Proposition

If two curvilinear polygons are quasi-isospectral, they have exactly the same quasi-eigenvalues.

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Let $\mathcal{P}$ and $\tilde{\mathcal{P}}$ be two quasi-isospectral admissible curvilinear polygons. Assume additionally that $\mathcal{P}$ is not exceptional. Then $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are loosely equivalent.

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Inverse Problem I: $\Sigma \to F$, recover a trigonometric polynomial by its roots

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Let $f : \mathbb{C} \to \mathbb{C}$ *be an even entire function of order one with a zero of order* $2m_0$ *at* $z = 0$*, and non-zero zeros* $\pm\gamma_j$ *repeated with multiplicities; denote by* Γ *the sequence (with multiplicities) consisting of* m_0 *zeros and* γ_j*.*

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Inverse Problem I: $\Sigma \to F$, recover a trigonometric polynomial by its roots

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$\inf\left\{r \in \mathbb{R} : f(z) = O\left(e^{|z|^r}\right) \text{ as } |z| \to \infty\right\} = 1$

Let $f : \mathbb{C} \to \mathbb{C}$ be an even entire function of order one with a zero of order $2m_0$ at $z = 0$, and non-zero zeros $\pm\gamma_j$ repeated with multiplicities; denote by Γ the sequence (with multiplicities) consisting of m_0 zeros and γ_j.

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$$f(z) = CQ_\Gamma(z), \quad Q_\Gamma(z) := z^{2m_0} \prod_{\gamma_j \in \Gamma \setminus \{0\}} \left(1 - \frac{z^2}{\gamma_j^2}\right).$$

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Hadamard-Weierstrass Theorem immediately gives the result with an extra factor $e^{g(z)}$, where $g(z)$ is linear. But since f is even, so is g, which is therefore a constant. □

Write

$$F_{\boldsymbol{\alpha},\boldsymbol{\ell}}(\sigma) = \sum_{k=1}^{\#\mathcal{T}} r_k \cos(t_k\sigma) - r_0, \qquad \mathcal{T} := \{|\boldsymbol{\ell}\cdot\boldsymbol{\zeta}| : \boldsymbol{\zeta} \in \mathfrak{Z}^n_+\}.$$

We want to find all t_k, r_k from the infinite product $Q_\Sigma(\sigma)$.

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Inverse Problem I: from an infinite product to the explicit form of a trigonometric polynomial

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Besicovitch mean of
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with C found from $2C\mathcal{A}[Q](\max\mathcal{T}) = 1$.

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Quasi-isospectral curvilinear polygons have the same quasi-eigenvalues

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If Λ is the spectrum of a curvilinear polygon $\mathcal{P}(\boldsymbol{\alpha}, \boldsymbol{\ell})$ then

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- Our statement requires a qualified convergence $\lambda_m - \sigma_m = O(m^{-\epsilon})$ as $m \to \infty$ rather than $o(1)$.

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- Proof is based on a technical bound $\lim\limits_{\sigma\to\infty} (Q_\Lambda(\sigma) - C_0 Q_\Sigma(\sigma)) = 0$ with some constant C_0.
- Allows the recovery of the frequencies and amplitudes of $F_{\boldsymbol{\alpha},\boldsymbol{\ell}}(\sigma)$ as before since $\mathcal{A}[f + o(1)](z) = \mathcal{A}[f](z)$ for all z.

Ideas of Proof

Inverse Problem III: $F \to \ell, \pm\mathbf{c}_{\boldsymbol{\alpha}}$, recover geometric information from a trigonometric polynomial

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$\boldsymbol{\ell}$ incommensurable over $\{-1, 1, 0\}$; no special angles

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immediately gives us the number of vertices n

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We will first find $\boldsymbol{\ell}'$ — the permutation of the vector of length in order of magnitude, $\ell'_1 < \ell'_2 < \cdots < \ell'_n$.

Easier to show on a concrete example. We will not need r_k's at this stage.

Ideas of Proof

Inverse Problem III: recover $\boldsymbol{\ell}'$

Example (we don't care about amplitudes for now; terms ordered by frequencies):

$$F(\sigma) = \sum_{j=1}^{8} ?\cos(t_j\sigma) - ? = \qquad \boxed{t_j \in \mathcal{T} = \{|\boldsymbol{\ell}\cdot\boldsymbol{\zeta}| : \boldsymbol{\zeta} \in \mathfrak{Z}_+^n\}}$$

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- Look for the next biggest frequency $t_7 = 19 = L - 2\ell'_1 = 23 - 2 \times 2$

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$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3,$$

- The next biggest frequency is $t_6 = 17 = L - 2\ell'_2 = 23 - 2 \times 3$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3,$$

- Remove all remaining frequencies in which either ℓ'_1 or ℓ'_2 or both come with a minus: $13 = 23 - 2 \times 2 - 2 \times 3$

Example (we don't care about amplitudes for now; terms ordered by frequencies):

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Ideas of Proof

Inverse Problem III: recover $\boldsymbol{\ell}'$

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3, 7,$$

- The biggest remaining frequency is $t_4 = 9 = L - 2\ell'_3 = 23 - 2\times 7$

Ideas of Proof

Inverse Problem III: recover $\boldsymbol{\ell}'$

Example (we don't care about amplitudes for now; terms ordered by frequencies):

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- Remove all remaining frequencies in which any of ℓ_1', ℓ_2', or ℓ_3' comes with a minus

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Eight terms, so $n = 4$.

$$L = 23 \qquad \boldsymbol{\ell}' = (2, 3, 7, 11)$$

remaining frequency is $t_1 = 1 = L - 2\ell'_4 = 23 - 2 \times 11$

Now we can look at the full polynomial

$$\begin{aligned} F(\sigma) &= \sum_{j=1}^{8} r_j \cos(t_j\sigma) - r_0 \\ &= \tfrac{1}{3}\cos(\sigma) - \tfrac{1}{60}\cos(3\sigma) + \tfrac{1}{8}\cos(5\sigma) + \tfrac{1}{10}\cos(9\sigma) \\ &\quad - \tfrac{2}{15}\cos(13\sigma) - \tfrac{1}{6}\cos(17\sigma) + \tfrac{1}{20}\cos(19\sigma) + \cos(23\sigma) + \tfrac{\sqrt{3}}{2\sqrt{2}} \end{aligned}$$

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Each of the frequencies t_j is written as a linear combination of ℓ'_k with $+$'s or $-$'s; write then

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Ideas of Proof

Inverse Problem III: recover $\boldsymbol{\ell}$ in proper order and $\mathbf{c}_{\boldsymbol{\alpha}}$

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Continuing …… — we are only interested in coefficients with one or two (or $n-1$, $n-2$) minuses

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Inverse Problem III: recover $\boldsymbol{\ell}$ in proper order and $\mathbf{c}_{\boldsymbol{\alpha}}$

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Write now the coefficients as a matrix,

$$R' = \left(R'_{p,q}\right)_{p,q,=1}^{n} = \begin{pmatrix} \frac{1}{20} & -\frac{2}{15} & \frac{1}{8} & -\frac{1}{60} \\ -\frac{2}{15} & -\frac{1}{6} & -\frac{1}{60} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{60} & \frac{1}{10} & -\frac{2}{15} \\ -\frac{1}{60} & \frac{1}{8} & -\frac{2}{15} & -\frac{1}{3} \end{pmatrix}$$

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We will now use this matrix to recover the correct order of sides and the cosine vector. We need to find the permutation (m_k) such that $\ell'_k = \ell_{m_k}$.

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Look at the off-diagonal elements of D'.

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Look at the off-diagonal elements of D'. If $D'_{k,j} \neq 1$, then ℓ'_k and ℓ'_j are neighbours, and

$$\left| \cos \frac{\pi^2}{2\alpha_{\ell'_k,\ell'_j}} \right| = \sqrt{D'_{k,j}} \qquad !$$

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The signs of diagonal elements allow us to find

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The signs of diagonal elements allow us to find $\mathbf{c}_{\boldsymbol{\alpha}} = \pm\left(\frac{1}{5}, \frac{1}{4}, -\frac{2}{3}, \frac{1}{2}\right)$.

A little bit of demystifying

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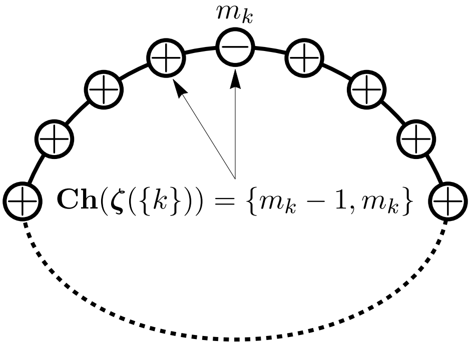
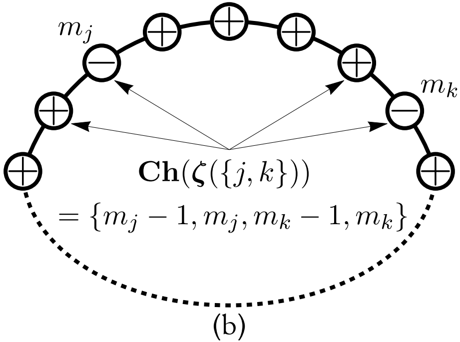


(a)

A little bit of demystifying



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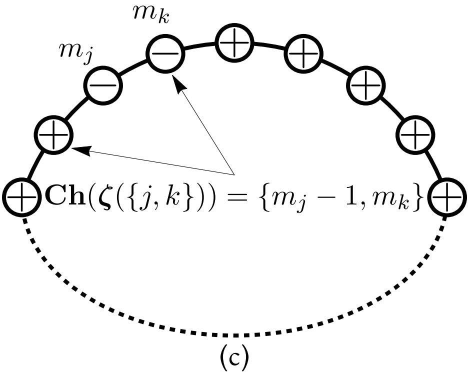

Modification of the Main Theorem in the presence of exceptional angles

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Theorem

Let $\mathcal{P}$ be an admissible curvilinear polygon.

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ℓ *incommensurable over* $\{-1, 1, 0\}$; *no special angles*

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- *The number K of exceptional components*

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Let $\mathcal{P}$ be an admissible curvilinear polygon. Then we can recover

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We may have

$$D' = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -\frac{1}{2} & \frac{1}{4} \\ 1 & 1 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Then $\mathcal{Y}_1 = (\ell_1')$, $\mathcal{Y}_2 = (\ell_2')$, $\mathcal{Y}_3 = (\ell_3', \ell_4')$.

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Two straight triangles with the same perimeter and angles $\boldsymbol{\alpha} = \left(\frac{\pi}{7}, \frac{\pi}{63}, \frac{53\pi}{63}\right)$ and $\tilde{\boldsymbol{\alpha}} = \left(\frac{\pi}{9}, \frac{\pi}{21}, \frac{53\pi}{63}\right)$ are quasi-isospectral.

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Example 3 — sides commensurable

A pair of curvilinear triangles with sides $\boldsymbol{\ell} = (3, 1, 1)$ and $\tilde{\boldsymbol{\ell}} = (2, 2, 1)$ and cosine vectors $\mathbf{c} = \left(\frac{1}{2}, \frac{1}{2}, \frac{-39+\sqrt{241}}{40}\right)$, $\tilde{\mathbf{c}} = \left(\frac{1}{2}, \frac{7-\sqrt{241}}{12}, \frac{-19+\sqrt{241}}{40}\right)$ are quasi-isospectral.