1. More on cuspidal representations

Definition 1.1. Let (π, V) be a smooth representation of G. Let $v \in V$ and $v^\vee \in V^\vee$. The matrix coefficient associated to $v, v^\vee$ is the function

$$\begin{aligned} c_{v,v^\vee} : G &\to \mathbb{C} \\ g &\mapsto \langle v^\vee, \pi(g)v \rangle. \end{aligned}$$

We say that (π, V) is Z-compact if every matrix coefficient is compactly supported modulo the centre $Z \subset G$.

Fact 1.2. Here are some important facts about cuspidal representations. We will not have time to prove them in this course, but the second and third rely on the first:

(1) A smooth representation of G is cuspidal if and only if it is Z-compact.
(2) An irreducible cuspidal representation of G is admissible.
(3) Let (π, V) be an irreducible cuspidal representation with central character ω_π. Then π is injective and projective in the category of smooth representations with central character ω_π.

We'll explain how to deduce (2) from (1). It suffices to show that for any non-zero $v \in V$, and $K \subset G$ compact open, the subspace of V spanned by $\{e_K \pi(g)v : g \in G\}$ is finite dimensional (since this subspace is $e_K V = V^K$). Extending this argument, you can also show that for a Z-compact representation, the function $g \mapsto e_K \pi(g)v$ is compactly supported mod Z for every v.

So suppose we have an infinite sequence $g_1, g_2, \dots$ such that the vectors $v_i := e_K \pi(g_i)v$ are linearly independent. We let $\lambda : V^K \to \mathbb{C}$ be a linear functional satisfying $\lambda(v_i) = 1$, and extend it to $\lambda : V \to \mathbb{C}$ satisfying $\lambda(v) = \lambda(e_K v)$. It follows that λ is fixed by K (acting via the dual representation), so $\lambda \in (V^\vee)^K$. The matrix coefficient $c_{v,\lambda}$ satisfies $c_{v,\lambda}(g_i) = 1$, so its support contains all of the g_i. This contradicts the assumption that it is compactly supported mod Z.

Corollary 1.3. *All irreducible smooth representations of G are admissible.*

Proof. Combine (2) in the above fact and Jacquet's subrepresentation theorem. □

Corollary 1.4. *An irreducible smooth representation is cuspidal if and only if it is not a subquotient of $n\text{-}\mathrm{Ind}_P^G W$ for any proper parabolic $P = LU \subset G$ and (for simplicity, irreducible, or more generally with a central character) smooth representation (σ, W) of L.*

This corollary says that 'cuspidal is equivalent to supercuspidal'.

Proof. It follows from Frobenius reciprocity that if π is not a subquotient of any parabolic induction then it is cuspidal. Conversely, suppose (π, V) is cuspidal, but that it does appear as a subquotient of a parabolic induction. More precisely, we have G-subspaces $V_2 \subset V_1 \subset n\text{-}\mathrm{Ind}_P^G W$ such that $V_1/V_2 \cong V$. The central character of $n\text{-}\mathrm{Ind}_{PG} W$ must match that of V. By (3) in the above fact, the surjective map $V_1 \to V$ has a G-equivariant splitting. This shows that $\mathrm{Hom}_G(V, n\text{-}\mathrm{Ind}_P^G W) \neq 0$, contradicting cuspidality of V. □

Example. Now we consider the Steinberg representation of $G = \mathrm{GL}_2(F)$: recall that we have a short exact sequence

$$0 \to \mathbb{C} \to n\text{-}\mathrm{Ind}_B^G \delta_B^{-1/2} \to St \to 0.$$

Computing the Jacquet module, we have $J_B\mathbb{C} = \delta_B^{-1/2}$, $J_B(n\text{-}\mathrm{Ind}_B^G \delta_B^{-1/2}) = \delta_B^{-1/2} \oplus \delta_B^{1/2}$, so $J_B St = \delta_B^{1/2}$. We know from Corollary 1.4 that every irreducible subquotient of St is non-cuspidal, so has non-zero J_B. This implies that St is irreducible.

Now we can consider the dual representation:

$$0 \to St^\vee \to n\text{-}\mathrm{Ind}_B^G \delta_B^{1/2} \to \mathbb{C} \to 0.$$

So we also have $J_B(St^\vee) = \delta_B^{1/2}$. By Frobenius reciprocity, $\mathrm{Hom}_G(St^\vee, n\text{-}\mathrm{Ind}_P^G \delta_B^{-1/2}) = 0$ and $\mathrm{Hom}_G(St, n\text{-}\mathrm{Ind}_P^G \delta_B^{1/2})$ is one-dimensional. We deduce that $St \cong St^\vee$ and the above short exact sequences are non-split.

Here is an important technical result which can be deduced from what we now know about Jacquet modules and cuspidal representations:

Theorem 1.5. *Any finitely generated admissible smooth representation of G has finite length.*

Proof. See Theorem 6.3.10 in Casselman. □

1.1. **Supercuspidal support.** We have shown that every irreducible smooth representation π of G appears as a subrepresentation of $n\text{-}\mathrm{Ind}_P^G W$ for a parabolic subgroup $P \subset G$ and an irreducible cuspidal representation (σ, W) of the Levi quotient L of P. The pair (L, σ) is referred to as the 'supercuspidal support' of π. It was proved by Bernstein and Zelevinsky that this supercuspidal support is essentially unique:

Theorem 1.6. *Let $P = LU$ and $P' = L'U'$ be two (standard) parabolic subgroups of G, and let (σ, W), (σ', W') be irreducible cuspidal representationsof L, L' respectively. Let $V = n\text{-}\mathrm{Ind}_P^G W$ and $V' = n\text{-}\mathrm{Ind}_P^G W'$. Then the following are equivalent:*

1. *There exists $w \in W$ such that $wMw^{-1} = M'$ and $W^w \cong W'$.*
2. $\mathrm{Hom}_G(V, V') = \mathrm{Hom}_{L'}(J_{P'}V, W') \neq 0$.
3. *V and V' have an isomorphic Jordan–Hölder constituent.*
4. *The Jordan–Hölder constituents of V and V' are isomorphic and appear with the same multiplicities.*

A very simple example of the above theorem is given by $G = \mathrm{GL}_2(F)$, $P = P' = B$, $\sigma = \delta_B^{1/2}$ and $\sigma' = \delta_B^{-1/2}$.

1.2. **Examples of supercuspidal representations.** There is a notion of cuspidal representations for finite groups of Lie type, such as $\mathrm{GL}_n(k)$ (recall that $k \cong \mathbb{F}_q$ is the residue field of F). We just ask that the space of coinvariants V_U vanishes for the unipotent subgroup U in any (standard) parabolic subgroup of $\mathrm{GL}_n(k)$. Since $\mathbb{C}$ representations of finite groups are completely reducible, we have $V_U \cong V^U$.

So now suppose (σ, W) is a cuspidal representation of $\mathrm{GL}_n(k)$. We can inflate it to a smooth representation of K_0. Let $\psi : Z \to \mathbb{C}^\times$ be a character of the centre in $G = \mathrm{GL}_n(F)$ which agrees with the central character of σ on $Z \cap K_0$. Then we can extend σ to a smooth representation (we again denote it by σ, but it depends on the choice of ψ) of the compact-mod-centre open subgroup $ZK_0 \subset G$.

In this section, we sketch the proof of:

Theorem 1.7. *The compact induction* $(\pi, V) = c\text{-}\mathrm{Ind}_{ZK_0}^G W$ *is an irreducible cuspidal representation of* G.

Proof. First we prove irreducibility. Suppose we have a proper G-subspace $V' \subset V$. By Frobenius reciprocity, we have $\mathrm{Hom}_{ZK_0}(V', W) = \mathrm{Hom}_G(V', \mathrm{Ind}_{ZK_0}^G W) \supset \mathrm{Hom}_G(V', c\text{-}\mathrm{Ind}_{ZK_0}^G W) \neq 0$.

By Frobenius reciprocity for compact inductions from open subgroups[1], we also have $\mathrm{Hom}_G(V, V/V') = \mathrm{Hom}_{ZK_0}(W, V/V')$. We deduce that W appears in both V' and V/V' as an irreducible constitutent. We will show that this is a contradiction.

We need to show that $V|_{ZK_0}$ contains W with multiplicity one. We can compute this restriction using the Mackey formula:

$$V|_{ZK_0} \cong \bigoplus_{g \in ZK_0 \backslash G / ZK_0} \mathrm{Ind}_{ZK_0 \cap g^{-1} ZK_0 g}^{ZK_0} (W^g)$$

the action of $g^{-1} ZK_0 g$ on W^g is given by letting $g^{-1}hg$ act as $\sigma(g)$.

Since $V|_{ZK_0}$ is semisimple (K_0 is compact and Z acts by a character), we need to show that $\mathrm{Hom}_{ZK_0 \cap g^{-1} ZK_0 g}(W, W^g) = 0$ for $g \in G - ZK_0$.

For (notational!) simplicity, we'll do this for $G = \mathrm{GL}_2$ (see Prasad–Raghuram for the general case). Using the Cartan decomposition, we can assume that $g = \mathrm{diag}(\varpi^m, 1)$, where $m > 0$. As usual, N is the upper triangular unipotent, and $N_0 = N \cap K_0$. We have $N_0 \cap g N_0 g^{-1} = g N_0 g^{-1} = \begin{pmatrix} 1 & \varpi^m \mathcal{O} \\ 0 & 1 \end{pmatrix}$, which necessarily acts trivially on W. So we have $\mathrm{Hom}_{N_0 \cap g N_0 g^{-1}}(W, W^g) \neq 0$ if and only if W^g has non-zero invariants under $g N_0 g^{-1}$, or equivalently if W has non-zero invariants under N_0. By our cuspidality assumption, $W^{N_0} = 0$ and we are done.

It remains to prove that V is cuspidal. To show that coinvariants V_U vanish for a parabolic $P = LU$, it suffices to prove that $\mathrm{Hom}_U(\pi, \mathbb{C}) = (\pi^*)^U = 0$. Note that here we are using the abstract vector space dual, not the smooth dual.

We can identify $(\pi^*)^U$ with functions $f : G \to W^*$ such that $f(kgu) = \sigma^*(k) f(g)$ for all $k \in ZK_0$ and $u \in U$. Since $G = BK_0 = K_0 P$ (Iwasawa decomposition), we can choose double coset representatives in L for $ZK_0 \backslash G / U$.

For $u \in U_0 = K_0 \cap U$ and $l \in L$, we have

$$\sigma^*(u) f(l) = f(ul) = f(l l^{-1} u l) = f(l).$$

So $f(l) \in (W^*)^{U_0}$. But cuspidality of σ implies that this space vanishes. ☐

[1] See exercise.

1.3. **Bernstein's second adjunction.** Recall that we had, as an immediate consequence of Frobenius reciprocity, an adjunction:

$$\mathrm{Hom}_G(V, \mathrm{Ind}_P^G W) = \mathrm{Hom}_L(V_U, W).$$

A much deeper theorem of Bernstein describes the *right* adjoint functor to parabolic induction. We let $\overline{P} = L\overline{U}$ be the opposite (lower triangular) parabolic to P.

Recall that in the geometric lemma we considered the P-equivariant map $\mathrm{Ind}_P^G W \to W$ given by mapping f to $f(1_G)$, which factors through $(\mathrm{Ind}_P^G W)_U$. Now we consider the map $W \to (\mathrm{Ind}_P^G W)_{\overline{U}}$ which takes w to the image of function supported on the open subset $P\overline{U} \subset G$ defined by $f(pu) = \sigma(p)w$. This is an L-equivariant map, and defines a natural transformation from the trivial functor to $(\mathrm{Ind}_P^G -)_{\overline{U}}$. In the usual way, this induces natural maps

$$\mathrm{Hom}(\mathrm{Ind}_P^G W, V) \to \mathrm{Hom}(W, V_{\overline{U}})$$

and Bernstein's theorem is that these maps are isomorphisms.

Corollary 1.8 (Uniform admissibility)**.**

(1) *J_P commutes with infinite direct products. In particular, any product of cuspidal representations is cuspidal.*

(2) *Let (π, V) be an irreducible cuspidal (equivalently, Z-compact) representation, and let $K \subset G$ be a compact open subgroup. There exists a constant $c(K)$ (independent of π!) such that* $\dim V^K < c(K)$.

Proof. The first part is category theory, since we have shown that J_P admits a left adjoint.

For the second part, suppose for a contradiction that $\pi_1, \pi_2, \pi_3, \ldots$ are a sequence of irreducible cuspidals with $\dim(V_i^K)$ non-zero and unbounded. Let v_i be a non-zero vector in each V_i^K. Consider the functions $f_i : g \mapsto e_K \pi_i(g) v_i$ and $f : g \mapsto e_K \prod \pi_i(g)(\prod v_i)$. We have $\mathrm{Supp}(f_i) \subset \mathrm{Supp}(f)$ for each i. On the other hand, the image of f_i generates V_i^K for each i, so $\dim V_i^K$ is at most the number of translates of ZK required to cover $\mathrm{Supp}(f_i)$. Since this number tends to infinity, we deduce that $\mathrm{Supp}(f)$ is not compact, which contradicts the cuspidality of $\prod_i V_i$. □

2. Exercise

2.1. **Frobenius reciprocity for compact inductions.** Show that if $H \subset G$ is an open subgroup of a locally profinite group, and $(\pi, V), (\sigma, W)$ are smooth representations of G, H respectively, then there are natural isomorphisms $\mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G W, V) = \mathrm{Hom}_H(W, V)$.