1. Parabolic induction

Let $n_1, n_2, \ldots, n_r$ be positive integers with $\sum n_i = n$. We denote by $P_{n_1,\ldots,n_r} \subset \mathrm{GL}_n(F)$ the standard parabolic subgroup: block upper triangular matrices with block size given by $(n_1, n_2, \ldots, n_r)$. For example

$$P_{2,1} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

We have $P = L \ltimes U$, where the Levi subgroup $L \cong \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$ and $U \triangleleft P$ is the unipotent subgroup, e.g.

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \subset P_{2,1}$$

Definition 1.1. If (σ, W) is a smooth representation of L, we obtain a smooth representation of P by inflation, and then a smooth representation of G by induction.

We again denote this representation by $\mathrm{Ind}_P^G W$.

Example. If $\chi_1, \ldots, \chi_n : F^\times \to \mathbb{C}^\times$ are locally constant characters, we obtain a character

$$\chi := \chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n : T \to \mathbb{C}^\times$$
$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto \prod_{i=1}^{n} \chi_i(t_i)$$

and hence a smooth representation $\mathrm{Ind}_B^G \chi$ of G.

A variant of the above definition is the *normalized induction* $n\text{-}\mathrm{Ind}_P^G W := \mathrm{Ind}_P^G \left(\delta_P^{1/2} W\right)$. It follows from our facts induced representations that the normalized induction interacts well with duality:

$$(n\text{-}\mathrm{Ind}_P^G W)^\vee \cong n\text{-}\mathrm{Ind}_P^G (W^\vee).$$

1.1. The Jacquet module.

Let $P \subset G = \mathrm{GL}_n(F)$ be a parabolic subgroup. Suppose we have a (smooth) P-representation (π, V). We denote by V_U the largest quotient of V on which U acts trivially (the U-coinvariants). Explicitly, we have

$$V_U = V / \langle \pi(u)v - v : u \in U, v \in V \rangle.$$

Since U is a normal subgroup of P, the P action on V naturally induces a smooth action of $L = P/U$ on V_U.

Proposition 1.2.

- *Taking coinvariants $V \mapsto V_U$ is an exact functor from smooth P-representations to smooth L-representations.*
- *(Frobenius reciprocity): For V, W smooth representations of G, L respectively, there are natural isomorphisms* $\mathrm{Hom}_G(V, \mathrm{Ind}_P^G W) = \mathrm{Hom}_L(V_U, W)$.

- *If V is an admissible smooth G-representation, V_U is an admissible smooth L-representation.*

As with induction, there is a normalized version of this construction, we will call this the Jacquet module: $J_P V = \delta_P^{-1/2} V_U$. Now Frobenius reciprocity takes the form:

$$\mathrm{Hom}_G(V, n\text{-}\mathrm{Ind}_P^G W) = \mathrm{Hom}_L(J_P V, W).$$

Theorem 1.3. *Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n : T \to \mathbb{C}^\times$ be a locally constant character as in the example above. Then the semisimplification of the Jacquet module $J_B(n\text{-}\mathrm{Ind}_B^G \chi)$ is isomorphic to $\bigoplus_{w \in W} \mathbb{C}(\chi^w)$.*

I called this 'Jacquet's geometric lemma' but apparently it should be attributed to Bernstein-Zelevinsky and Casselman

In the theorem, the sum is over the Weyl group, and the character $\chi^w : T \to \mathbb{C}^\times$ is defined by $\chi^w(t) = \chi(wtw^{-1})$.

Proof. We'll explain this for $G = \mathrm{GL}_2(F)$. See Prasad's notes for an outline of the proof for GL_n. The map given by evaluating functions at 1_G gives a surjective B-equivariant map

$$n\text{-}\mathrm{Ind}_B^G \chi \to \mathbb{C}(\delta_B^{1/2}\chi).$$

The kernel of this map is given by the subspace of functions in $n\text{-}\mathrm{Ind}_B^G \chi$ which vanish at 1_G (and therefore vanish on all of B). We denote this space by W. So we have a (B-equivariant) short exact sequence

$$0 \to W \to n\text{-}\mathrm{Ind}_B^G \chi \to \mathbb{C}(\delta_B^{1/2}\chi) \to 0$$

and taking Jacquet modules we get a short exact sequence

$$0 \to J_B W \to J_B(n\text{-}\mathrm{Ind}_B^G \chi) \to \mathbb{C}(\chi) \to 0$$

Recall that $G = B \coprod BwB$. The map

$$f \mapsto [n \mapsto f(wn)]$$

defines an N-equivariant isomorphism of $\mathbb{C}$-vector spaces between W and $\mathcal{C}_c^\infty(N)$ (N acts on the target via right translation on itself). We want to compute $J_B W$.

Exercise 2.2 shows that $J_B W = W/\ker(\phi)$, where $\phi(f) = \int_N f(wn)dn$. This shows that $J_B W$ is a one-dimensional space, and we compute the action of T by considering $\phi(tf) = \int_N f(wnt)dn = \int_N f(wtw^{-1}wt^{-1}nt)dn = \chi^w(t)\delta_B^w(t)^{1/2} \int_N f(wt^{-1}nt)dn = \chi^w(t)\delta_B(t)^{-1/2} \int_N f(wt^{-1}nt)dn = \chi^w(t)\delta_B(t)^{-1/2}\delta_B(t)\phi(f)$. □

Example. We consider $G = \mathrm{GL}_2(F)$ and χ a character as in the above theorem. Then if $\chi \neq \chi^w$ we can automatically conclude that $J_B(n\text{-}\mathrm{Ind}_B^G \chi) = \chi \oplus \chi^w$. If $\chi = \chi^w$ it turns out that the Jacquet module is not semisimple.

Corollary 1.4. *Suppose $G = \mathrm{GL}_2(F)$ and χ is a unitary locally constant character of T (in other words, $\chi = \chi_1 \otimes \chi_2$ and $|\chi_1(\varpi)| = |\chi_2(\varpi)| = 1$). Then $n\text{-}\mathrm{Ind}_B^G \chi$ is irreducible and $n\text{-}\mathrm{Ind}_B^G \chi \cong n\text{-}\mathrm{Ind}_B^G \chi^w$.*

Proof. We have $(n\text{-Ind}_B^G\chi)^\vee \cong n\text{-Ind}_B^G\chi^{-1}$ and since χ is unitary, χ^{-1} is the complex conjugate of χ. It follows that we can equip $n\text{-Ind}_B^G\chi$ with a G-invariant Hermitian inner product (we say that the representation is unitary). More explicitly, this inner product is given by

$$\langle f, g\rangle = \int_{K_0} f(k)\overline{g(k)}dk.$$

Admissible unitary representations are semisimple (see exercise 2.1). It follows from Frobenius reciprocity and the above example that

$$\mathrm{Hom}_G(n\text{-Ind}_B^G\chi, n\text{-Ind}_B^G\chi) = \mathrm{Hom}_T(J_B\left(n\text{-Ind}_B^G\chi\right), \chi)$$

is one-dimensional, so we deduce that $n\text{-Ind}_B^G\chi$ is irreducible. For the final part, we again apply Frobenius reciprocity, this time to compute $\mathrm{Hom}_G(n\text{-Ind}_B^G\chi, n\text{-Ind}_B^G\chi^w)$. □

1.2. **Reducibility in the non-unitary case.**

Theorem 1.5. *Suppose $G = \mathrm{GL}_2(F)$, and let χ be an arbitrary locally constant character of T. Then $n\text{-Ind}_B^G\chi$ is reducible if and only if $(\chi_1/\chi_2)(x) = |x|^{\pm 1}$.*

1.3. **Supercuspidal representations.**

Definition 1.6. We say that a smooth representation (π, V) of G is *cuspidal* if $J_P V = 0$ for every proper parabolic subgroup P.

We can extend this definition to representations of more general groups: for us, it is enough to just consider Levi subgroups L which are products of groups $GL_n(F)$.

Lemma 1.7. *Suppose we have two parabolic subgroups $P' \subset P \subset G$ and a smooth representation (π, V) of G. Then there are natural isomorphisms*

$$J_{P'\cap L}J_P V \cong J_{P'}V.$$

Lemma 1.8. *A smooth representation (π, V) is cuspidal if and only if*

$$\mathrm{Hom}_G(V, n\text{-Ind}_P^G W) = 0$$

for all (proper) parabolic subgroups $P \subset G$ and smooth representations (σ, W) of L.

Proof. Apply Frobenius reciprocity. □

Theorem 1.9 (Jacquet subrepresentation theorem)**.** *If (π, V) is an irreducible smooth representation of G, there exists a parabolic subgroup $P = LU$ and an irreducible cuspidal representation (σ, W) of L such that V is a subrepresentation of $n\text{-Ind}_P^G W$.*

Proof. Choose P minimal such that $J_P V \neq 0$ (if V is cuspidal we take $W = V$ and $P = G$). It can be checked that J_P preserves finite generation, so $J_P V$ is a finitely generated smooth representation of L. It therefore has an irreducible quotient (σ, W) (use Zorn's lemma to finite a maximal proper L-stable subspace). Transitivity of the Jacquet module implies that (σ, W) is supercuspidal. Frobenius reciprocity tells us that

$$0 \neq \mathrm{Hom}_L(J_P V, W) = \mathrm{Hom}_G(V, n\text{-Ind}_P^G W).$$

□

2. Exercises

2.1. **Semisimplicity of admissible unitary representations.** Suppose (π, V) is an admissible smooth representation of a locally profinite group G, equipped with a G-invariant Hermitian inner product $\langle , \rangle$. Show that π is semisimple; more precisely, for every G-stable subspace $W \subset V$, define $W^\perp = \{v \in V : \langle v, w\rangle = 0 \text{ for all } w \in W\}$ and prove that $V = W \oplus W^\perp$.

Hint: Show that $\langle , \rangle$ defines an inner product on V^K for any compact open subgroup $K \subset G$.

2.2. **More on the Jacquet module.** Let $P \subset G = \mathrm{GL}_n(F)$ be a parabolic subgroup and let (V, π) be a smooth P-representations. Recall that for $P = LU$ the Jacquet module is defined in terms of the coinvariants:

$$V_U = V/\langle \pi(u)v - v : u \in U, v \in V\rangle.$$

We let $V(U) = \langle \pi(u)v - v : u \in U, v \in V\rangle$. If $U_c \subset U$ is a compact open subgroup, we define $\pi_{U_c} : V \to \mathbb{C}$ by

$$\pi_{U_c}(v) = \int_{U_c} \pi(u)v du.$$

In other words, we apply the Hecke operator $e_{U_c} \in \mathcal{H}(U)$.

Show that $V(U) = \{v \in V : \exists U_c \text{ such that } \pi_{U_c}(v) = 0\}$, and use this to prove exactness of taking coinvariants.

Hint: first show that U is equal to the union of its compact open subgroups, so $V(U) = \cup_{U_c} V(U_c)$. Then show that $V(U_c) = \ker \pi_{U_c}$.

2.3. **Explicit description of principal series for GL_2.**

Show that the map

$$n\text{-}\mathrm{Ind}_B^G \chi \to \mathcal{C}^\infty(F) \oplus \mathbb{C}$$
$$f \mapsto \left(x \mapsto f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right), f(1_G)\right)$$

is injective, with image the pairs (ψ, y) such that there exists an $r > 0$ such that

$$\psi(x) = (\delta_B^{1/2}\chi)\begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} \cdot y$$

for all $x \in F$ with $|x| \geq r$.

Hint: there exists an r such that $f(1_G) = f\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}$ for all $x \in F$ with $|x| \geq r$.