1. More preliminaries on representations of $\mathrm{GL}_n(F)$

The example $g \mapsto \begin{pmatrix} 1 & \log|\det(g)| \\ 0 & 1 \end{pmatrix}$ shows that not all smooth representations are semisimple.

1.1. **Haar measure.**

It will be useful to think about integrating functions on G (and other locally profinite groups).

We let H be an arbitrary locally profinite group. The spaces of functions we'll be interested in are locally constant, compactly supported functions $f : H \to \mathbb{C}$. We denote this space of functions by $\mathcal{C}_c^\infty(H)$. Locally constant means that for every $g \in H$ there is an open neighbourhood $g \in U$ such that f is constant on U. Compact support means that the closure of the set $\{g \in H : f(g) \neq 0\}$ is compact. You can check that for $f \in \mathcal{C}_c^\infty(H)$ there is a compact open $K \subset H$ such that $f(gk) = f(g)$ for all $g \in H$ and $k \in K$.

It's not hard to see that if $f \in \mathcal{C}_c^\infty(H)$, then there exists a compact open subgroup $K \subset G$, a finite collection of distinct cosets $(g_iK)_{i=1}^r$ in G/K and constants $c_i \in \mathbb{C}$ such that

$$f = \sum_{i=1}^r c_i \mathbf{1}_{g_iK}.$$

Our measures will be linear functionals $\mu : \mathcal{C}_c^\infty(H) \to \mathbb{C}$. It follows from the previous paragraph that they are uniquely determined by the values $\mu(gK) := \mu(\mathbf{1}_{gK})$, where g runs over H and K runs over compact open subsets of H. We will demand that the volumes $\mu(gK)$ are non-negative real numbers.

There are left and right actions of H on $\mathcal{C}_c^\infty(H)$:

$$h *_\lambda f : g \mapsto f(h^{-1}g)$$
$$h *_\rho f : g \mapsto f(gh).$$

Definition 1.1. A left Haar measure is a non-zero measure μ on H with $\mu(h *_\lambda f) = \mu(f)$ for all $h \in H$. You can guess what a right Haar measure is.

If we don't specify otherwise, when we say Haar measure we mean a left Haar measure.

It follows easily from our observation on elements in $\mathcal{C}_c^\infty(H)$ that a (left) Haar measures exists, and is unique up to a positive real scalar. For $G = GL_n(F)$, we will fix μ_G to be the Haar measure with $\mu_G(K_0) = 1$. If $K \subset K_0$ is compact open, then $\mu_G(K) = [K_0 : K]^{-1}$.

In general, if μ is a left Haar measure then μ may not be right-invariant. We can consider the measure $\mu^g : f \mapsto \mu(g *_\rho f)$. This is a left Haar measure, so we have $\delta(g) \in \mathbb{R}_{>0}$ with $\mu^g = \delta(g)\mu$. This defines a homomorphism $\delta : H \to \mathbb{R}_{>0}$.

Example.
1. $G = \mathrm{GL}_n(F)$ is *unimodular*, i.e. μ_G is a left and right Haar measure.
2. Consider $B \subset \mathrm{GL}_2(F)$. Then $\delta\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = |a/c|$, where the absolute value is normalized by $|\varpi| = |k_F|^{-1}$.

1.2. **Hecke algebras.** In this section G can be an arbitrary locally profinite group. Note that some references assume G is unimodular. We denote by $\mathcal{H}(G)$ the space of functions $\mathcal{C}_c^\infty(G)$ equipped with the *convolution product*:

$$(f * f')(x) = \int_G f(g)f'(g^{-1}x)d\mu_G.$$

This is associative (exercise!), not commutative in general, and has no unit if G is not compact. It does have lots of idempotents: suppose $K \subset G$ is compact open and let $e_K = \frac{1}{\mu_G(K)}\mathbf{1}_K$. Then $e_K * e_K = e_K$.

So we have a unital subalgebra $\mathcal{H}(G,K) = e_K\mathcal{H}(G)e_K \subset \mathcal{H}(G)$, with unit e_K.

The subalgebra $\mathcal{H}(G,K)$ is given by functions which are 'bi-K-invariant', i.e. $f(k_1gk_2) = f(g)$ for all $k_1, k_2 \in K$. The compact support condition implies that we can think of these as finitely supported functions on the set of double cosets $K\backslash G/K$.

If (V,π) is a smooth representation of G we can define an action of $\mathcal{H}(G)$ on V by:

$$\pi(f)v = \int_G f(g)\pi(g)vd\mu(g).$$

Remember that all of our integrals are secretly finite sums: if K is a compact open subgroup which fixes v and such that $f(gk) = f(g)$ for all $k \in K$ (such a K exists!) then we have

$$\pi(f)v = \sum_{g\in G/K} \mu_G(K)f(g)\pi(g)v$$

and this sum only has finitely many non-zero terms because f has compact support.

The $\mathcal{H}(G)$-module V inherits a smoothness property from the smoothness of π. We say that a $\mathcal{H}(G)$ module v is smooth if every $v \in V$ is fixed by e_K for some compact open subgroup K.

Theorem 1.2. *The above construction gives an equivalence of categories between smooth G-representations and smooth $\mathcal{H}(G)$-modules.*

Fix a compact open subgroup K. The map $V \mapsto V^K$ induces a bijection between isomorphism classes of irreducible smooth representations with $V^K \neq 0$ and isomorphism classes of simple $\mathcal{H}(G,K)$-modules.

Proof. The first part is a formality (if V is a smooth $\mathcal{H}(G)$-module then for $v \in V$ with $e_K v = v$ we can define $\pi(g)v = \frac{1}{\mu_G(K)}\mathbf{1}_{gK}v$). For the second part, it's not hard to check that if V is irreducible then V^K is a simple $\mathcal{H}(G,K)$-module. If M is a simple $\mathcal{H}(G,K)$-module, define a $\mathcal{H}(G)$-module $\tilde{M} = U/X$ where $U = \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M$ and X is the largest $\mathcal{H}(G)$-submodule of U with $e_K X = 0$ (equivalently, the largest $\mathcal{H}(G)$-submodule with $X \cap e_K U = 0$). It can be checked that $\tilde{M}$ is simple. If we start with a simple $\mathcal{H}(G)$-module V with $V^K \neq 0$, then the canonical map $\mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} V^K \to V$ is surjective and maps the submodule X to 0, so it induces an isomorphism $\tilde{V^K} \cong V$. $\square$

For general K this theorem is perhaps not so useful, but it is very important in the case where $G = \mathrm{GL}_n(F)$ and $K = K_0$ (and other special cases). We can already see that the

Cartan decomposition tells us something about $\mathcal{H}(G, K_0)$. We will compute this Hecke algebra explicitly later, and show it is commutative.

Now seems as good as time as any to record:

Definition 1.3. A smooth representation (π, V) of a locally profinite group H is *admissible* if V^K is finite dimensional for all compact open subgroups K.

An important fact (which requires more development of the representation theory to prove) is that all irreducible smooth representations of $G = \mathrm{GL}_n(F)$ are admissible (this is true more generally for p-adic reductive groups, but not for arbitrary locally profinite groups).

2. Parabolic induction

Here is an example.

Consider $\mathbb{P}^1(\mathbb{Q}_p)$ with its right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ (right multiplication on row vectors). Then we have a natural left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on the space of locally constant $\mathbb{C}$-valued functions $\mathcal{C}^\infty(\mathbb{P}^1(\mathbb{Q}_p))$. This is a smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. We have an inclusion of the trivial representation $\mathbb{C} \to \mathcal{C}^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ given by the constant functions. We will see that the cokernel of this inclusion is an irreducible (infinite-dimensional) smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. It is called the *Steinberg representation*.

2.1. **Induction.** Let G be a locally profinite group, let $H \subset G$ be a closed subgroup, and let (σ, W) be a smooth representation of H.

Definition 2.1. We define the space $\mathrm{Ind}_H^G W$ of functions $f : G \to W$ satisfying:

- for all $h \in H$ and $g \in G$, $f(hg) = \sigma(h)g(f)$,
- there exists an open subgroup $K \subset G$ (it can depend on f) such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$.

We equip this by the (left) action of G given by right translation, $\pi(g)f : g' \mapsto f(g'g)$.

The second condition on the functions immediately tells us that $\mathrm{Ind}_H^G W$ is a smooth representation of G.

We also define a sub-representation $c\text{-}\mathrm{Ind}_H^G W \subset \mathrm{Ind}_H^G W$ given by functions with compact support modulo H (i.e. there is a compact subset $\Omega \subset G$ with the support of the function contained in $H\Omega$). If $H\backslash G$ is compact, then $c\text{-}\mathrm{Ind}_H^G W = \mathrm{Ind}_H^G W$.

Here are some facts about induction:

(1) Ind_H^G and $c\text{-}\mathrm{Ind}_H^G$ define exact functors from smooth representations of H to smooth representations of G (it should be clear how to map a morphism of representations to a morphism of the induced representations).
(2) Ind and c-Ind are transitive: if $H \subset H' \subset G$ then $\mathrm{Ind}_{H'}^G \mathrm{Ind}_H^{H'} = \mathrm{Ind}_H^G$.
(3) (Frobenius reciprocity) The natural map $f \mapsto f(1_G)$ induces isomorphisms
$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) = \mathrm{Hom}_H(V, W)$$
for any representation V of G.

(4) Write δ_H, δ_G for the modulus characters of H and G. There is an isomorphism of G-representations

$$c\text{-Ind}_H^G(W)^\vee \cong \text{Ind}_H^G((\delta_G/\delta_H)W^\vee).$$

(5) If $H\backslash G$ is compact, Ind_H^G preserves admissibility.

(2), (3) and (5) are quite straightforward. Proofs of (1) and (4) can be found in Bernstein-Zelevinsky. The main ingredient for (1) is the following:

Let K be a compact open subgroup of G and fix a set Ω of representatives for the double cosets $H\backslash G/K$. For each $g \in \Omega$ set $K_g = H \cap gKg^{-1}$. Then the map

$$\begin{aligned}(\text{Ind}_H^G W)^K &\to \prod_{g\in\Omega} W^{K_g}\\ f &\mapsto (f(g))_{g\in\Omega}\end{aligned}$$

is an isomorphism (of vector spaces).

If we restrict to $(\text{Ind}_H^G W)^K$ we obtain an isomorphism with $\bigoplus_{g\in\Omega} W^{K_g}$.

This is combined with the fact that taking invariants under K (or K_g) is exact (since it is given by applying an idempotent in the Hecke algebra).

2.2. **Parabolic induction.**

3. Exercises

3.1. **Dual/contragredient representation.** Let (π, V) be a smooth representation of a locally profinite group H. Let $K \subset H$ be a compact open subgroup. Define the map

$$\begin{aligned}e_K : V &\to V^K\\ v &\mapsto \int_K \pi(k) v d\mu_K(k)\end{aligned}$$

where μ_K is the Haar measure on K with $\mu_K(K) = 1$ (note that K is unimodular).

(1) Show that e_K gives a splitting to the natural inclusion $V^K \hookrightarrow V$.

Let (π^*, V^*) be the dual representation. Let $V^\vee$ be the subset of *smooth* vectors in V^*: those $v \in V^*$ such that $\text{Stab}_H(V)$ is open.

(2) Show that π^* restricts to a smooth representation of H on $V^\vee$.

We denote this restriction by $\pi^\vee$. This is the *smooth dual* or *contragredient dual* representation of π. We will usually just say 'dual' because all our representations will be smooth.

(3) Show that if π is admissible, then $\pi^\vee$ is admissible and the natural map $V \to (\pi^*)^*$ induces an isomorphism $V \cong (\pi^\vee)^\vee$.

Hint: show that $(V^\vee)^K$ *can be identified with the dual of* V^K*, for any compact open* K*.*

3.2. **Double coset operators.** Let K be a compact open subgroup of G, let $g \in G$ let $[KgK] \in \mathcal{H}(G, K)$ be the indicator function of the double coset KgK.

If (V, π) is a smooth G-representation and $v \in V^K$, show that $\pi([KgK])v = \mu_G(K) \sum_{i=1}^r \gamma_i v$ where $KgK = \coprod_{i=1}^r \gamma_i K$.

Similarly, show that if $f \in \mathcal{H}(G, K)$ then $[KgK] * f = \mu_G(K) \sum_{i=1}^r \gamma_i *_\lambda f$.

3.3. **Commutativity of spherical Hecke algebra.** Let $G = \mathrm{GL}_n(F)$.

(1) Show that the map $\iota : \mathcal{H}(G, K_0) \to \mathcal{H}(G, K_0)$ defined by $\iota(f)(g) = f(g^t)$ (the transpose) satisfies $\iota(f * f') = \iota(f') * \iota(f)$.
(2) Use the Cartan decomposition to show that ι is the identity, and therefore $\mathcal{H}(G, K_0)$ is commutative.

3.4. **Spherical Hecke algebra for GL_2.** Let $G = \mathrm{GL}_2(F)$ and write $T_r = [K_0 \begin{pmatrix} \varpi^r & 0 \\ 0 & 1 \end{pmatrix} K_0]$ (for $r \geq 1$), $S = [K_0 \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} K_0] \in \mathcal{H}(G, K_0)$.

Show that $T_1 * T_r = T_{r+1} + |\varpi|^{-1} S * T_{r-1}$.

Hint: using the Cartan decomposition, to identify a Hecke operator you just need to check its values on elements of the form $\begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix}$.