1. p-adic numbers and p-adic groups

$\mathbb{Q}_p$: completion of $\mathbb{Q}$ with respect to metric induced by non-Archimedean absolute value $|p^n a/b| = p^{-n}$, a, b coprime to p, $n \in \mathbb{Z}$.

$\mathbb{Q}_p$ is a topological field. It is *locally profinite*:

Definition 1.1. A topological group G is locally profinite if it is Hausdorff and every open neighbourhood of the identity contains a compact open subgroup (equivalently, Hausdorff, locally compact and totally disconnected).

Fact 1.2. (See Gruenberg's article in Cassels–Fröhlich) A Hausdorff topological group G is profinite if and only if it is compact and totally disconnected.

More generally, we can consider p-adic local fields: finite extension fields $F/\mathbb{Q}_p$, with ring of integers $\mathcal{O} = \mathcal{O}_F \subset F$, and maximal ideal $\mathfrak{m}_{\mathcal{O}} = (\varpi)$ generated by a uniformiser $\varpi \in F$.

In this course, we will interested in representations of the groups $\mathrm{GL}_n(F)$. Many things generalize easily (and some less easily) to the F-points of reductive linear algebraic groups G/F.

We equip $\mathrm{GL}_n(F) \subset M_n(F)$ with the subspace topology induced by the product topology on $M_n(F) \cong F^{\oplus n^2}$. This makes $\mathrm{GL}_n(F)$ into a topological group. We will often just write G for $\mathrm{GL}_n(F)$. We can check it is locally profinite. Here are some special compact open subgroups:

Definition 1.3. Set $K_0 = \mathrm{GL}_n(\mathcal{O}) \subset G$, $K_r = \{g \in K_0 : g \equiv I_n \mod \varpi^r\}$.

Then $(K_r)_{r\geq 0}$ is a basis of open neighbourhoods on the identity in G, in particular G is locally profinite.

1.1. Structure of G.

There are some important structure theorems for G.

1.1.1. *Cartan decomposition.*

Let $\Lambda = \{\mathrm{diag}(\varpi^{m_1}, \varpi^{m_2}, \dots, \varpi^{m_n}) : m_i \in \mathbb{Z}\}$ and $\Lambda^+ = \{\mathrm{diag}(\varpi^{m_1}, \varpi^{m_2}, \dots, \varpi^{m_n}) : m_1 \geq \dots \geq m_n\}$.

Then $G = K_0 \Lambda^+ K_0 = \coprod_{\lambda \in \Lambda^+} K_0 \lambda K_0$.

Proof. This is the 'Smith normal form'. It is a consequence of the structure theorem for finitely generated modules over PIDs. □

Consequence: G/K_0 is countable.

Real analogue: every $g \in \mathrm{GL}_n(\mathbb{R})$ has a polar decomposition $g = qu$, q a (symmetric) positive definite matrix ($q = \sqrt{m^t m}$) and $u \in O(n)$. We can furthermore diagonalise q with an orthogonal matrix.

1.1.2. *Iwasawa decomposition.*

$G = BK_0$, where $B \subset G$ is the Borel subgroup of upper triangular matrices.

Proof. Exercise for $n = 2$. □

1.1.3. *Bruhat decomposition.* $W = N_G(T)/T \cong S_n$ Weyl group ($T \subset G$ diagonal matrices). We have $G = BWB = \coprod_{w \in W} BwB$.

Proof. For $n = 2$, $W = \{1, w\}$, we are decomposing $G = B \coprod BwB$, you can check by hand that BwB gives all matrices with non-zero bottom left entry.

In general this decomposition is best understood in terms of the flag variety G/B, which parameterizes filtrations

$$\mathscr{F} : 0 = \mathscr{F}_0 \subset cF_1 \subset \cdots \subset \mathscr{F}_n = F^n$$

with $\dim_F \mathscr{F}_i = i$. The coset gB corresponds to the flag given by $\mathscr{F}_i = \langle ge_1, \ldots, ge_i \rangle$ with $e_1, \ldots, e_n$ the standard basis of F^n. This flag is stabilized by the Borel subgroup gBg^{-1}. The 'standard flag' is $\mathscr{F}_{std} : 0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \ldots$. We can measure the relative position of $\mathscr{F}$ and $\mathscr{F}_{std}$ by a permutation: there is a unique element $\sigma \in S_n$ such that there exists a basis $f_1 \ldots, f_n$ of F^n with $\mathscr{F}_{std,i} = \langle f_{\sigma 1} \ldots, f_{\sigma i} \rangle$ and $\mathscr{F}_i = \langle f_1, \ldots, f_i \rangle$ for every i. If M_f is the change of basis matrix corresponding to $f_1, \ldots, f_n$ then $g = M_f b$ for some $b \in B$ (since M_f and g give the same point of G/B). We also have $M_f \sigma \in B$, since $M_f \sigma$ is the change of basis matrix for $f_{\sigma 1} \ldots, f_{\sigma n}$. We deduce that $g \in B\sigma^{-1}B$. □

1.2. **Smooth representations.** Now we are going to define the kinds of representations of locally profinite groups we will be studying in this course. If H is any topological group and V is a $\mathbb{C}$-vector space equipped with a representation $\pi : H \to \mathrm{GL}(V)$, we say that V is *smooth* if for every $v \in V$ the stabilizer $\mathrm{Stab}_H(v)$ is open in H.

Example. If $\chi : F^\times \to \mathbb{C}^\times$ is a locally constant character, then χ defines a smooth representation of $\mathrm{GL}_1(F)$. Composing χ with the determinant, we obtains a smooth (one-dimensional) representation of $\mathrm{GL}_n(F)$ for any $n \geq 1$.

Note that χ being locally constant is equivalent to $\chi|_{\mathcal{O}_F^\times}$ factoring through the finite quotient $(\mathcal{O}/\varpi^r)^\times$ for some r. We will see some more interesting examples of smooth representations later!

Proposition 1.4 (Schur's lemma). *Suppose (V, π) is an irreducible smooth representation of G. If $\phi \in \mathrm{End}_G(V)$ then ϕ is multiplication by a scalar.*

Proof. First we need to find a singular value for ϕ. Suppose that $\phi - \lambda I_V$ is invertible for all $\lambda \in \mathbb{C}$. Then we get a map of $\mathbb{C}$-algebras

$$\begin{aligned} \mathbb{C}(X) &\to \mathrm{End}_G(V) \\ X &\mapsto \phi. \end{aligned}$$

The source is a $\mathbb{C}$ vector space of uncountable dimension. The target has countable dimension: if $v \in V - \{0\}$ then there is a compact open subgroup $K \subset G$ fixing v and V is generated by the countably many vectors $\pi(g)v$ as g runs over G/K. OK, so we conclude that there is a $\lambda \in \mathbb{C}$ with $\phi - \lambda$ not invertible. Either it has a non-zero kernel or its image is a proper subspace in V. In either case, it follows from irreducibility that $\phi - \lambda = 0$. □

Exercise: check G/K is countable using the Cartan decomposition. Later we will introduce *admissible* smooth representations; you can prove Schur's lemma for admissible things without using the countability argument.

Corollary 1.5 (Central character)**.** *Let $Z \subset G$ denote the scalar matrices. If (V, π) is an irreducible smooth representation of G, then there is a locally constant character $\omega_\pi : Z \to \mathbb{C}^\times$ such that $\pi(z) = \omega_\pi(z) I_V$ for all $z \in Z$.*

Proposition 1.6. *Let (π, V) be a finite dimensional irreducible smooth representation of $\mathrm{GL}_n(F)$. Then $\dim V = 1$ and $\pi(g) = \chi \circ \det(g)$ for some locally constant $\chi : F^\times \to \mathbb{C}^\times$.*

Proof. The essential point is to show that $\mathrm{SL}_n(F)$ acts trivially on V. So the representation factors through the determinant. Then use Schur's lemma. To show that $\mathrm{SL}_n(F)$ acts trivially, we use the fact that if V is finite dimensional then there is a single compact open subgroup K which acts trivially on V. In particular the kernel of π is an open normal subgroup of $\mathrm{GL}_n(F)$. Now there should be an argument using simplicity of $\mathrm{PSL}_n(F)$.

Alternatively, we can use the fact that $\mathrm{SL}_n(F) = \langle N, \overline{N} \rangle$ where $N \subset \mathrm{GL}_n(F)$ is the subgroup of upper triangular matrices with 1's on the diagonal and $\overline{N}$ is the lower triangular analogue. Since $\overline{N}$ is a conjugate of N, it suffices to show that N acts trivially on V. Since V is finite dimensional and smooth, there exists a single open subgroup K which acts trivially on all of V. For each element $n \in N$ and each $r \geq 0$, there is an element $t \in \mathrm{GL}_n(F)$ with $tnt^{-1} \in K_r$ (in fact, we can take $t \in \Lambda$). We can therefore choose t so that $tnt^{-1} \in K$. Then tnt^{-1} acts trivially on V, so n does. □

2. Exercises

2.1. **Maximal compact.** Let K be a compact subgroup of $G = \mathrm{GL}_n(F)$. Show that there exists $g \in G$ such that $gKg^{-1} \subset \mathrm{GL}_n(\mathcal{O})$. Show that $\mathrm{GL}_n(\mathcal{O})$ is maximal compact (i.e. any compact subgroup of G which contains $\mathrm{GL}_n(\mathcal{O})$ is equal to $\mathrm{GL}_n(\mathcal{O})$).

Hint: show that a compact subgroup stabilises an $\mathcal{O}$-lattice in F^n.

2.2. **Compact groups and admissibility.** Here is an important definition:

Definition. A smooth representation (π, V) of a topological group H is *admissible* if V^K is finite dimensional for all open subgroups $K \subset H$.

(1) Let K be a profinite group. Show that an irreducible smooth representation of K is finite dimensional.
(2) Let (π, V) be a smooth representation of a locally profinite group H, and let $K \subset H$ be any compact open subgroup. Show that π is admissible if and only if the restriction $\pi|_K$ is a direct sum of irreducibles, each occurring with finite multiplicity.