PARALLEL WEIGHT 2 POINTS ON HILBERT MODULAR EIGENVARIETIES AND THE PARITY CONJECTURE

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ABSTRACT. Let F be a totally real field and p an odd prime which is totally split in F. We define and study one-dimensional 'partial' eigenvarieties interpolating Hilbert modular forms over F with weight varying only at a single place v above p. For these eigenvarieties, we show that methods developed by Liu, Wan and Xiao apply and deduce that, over a boundary annulus in weight space of sufficiently small radius, the partial eigenvarieties decompose as a disjoint union of components which are finite over weight space. We apply this result to prove the parity version of the Bloch–Kato conjecture for finite slope Hilbert modular forms with trivial central character (under some assumptions), by reducing to the case of parallel weight 2. As another consequence of our results on partial eigenvarieties, we show, still under the assumption that p is totally split in F, that the 'full' (dimension $1 + [F : \mathbb{Q}]$) cuspidal Hilbert modular eigenvariety has the property that many (all, if $[F : \mathbb{Q}]$ is even) irreducible components contain a classical point with non-critical slopes and parallel weight 2 (with some character at p whose conductor can be explicitly bounded), or any other algebraic weight.

1. INTRODUCTION

1.1. **Eigenvarieties near the boundary of weight space.** In recent work, Liu, Wan and Xiao [LWX] have shown that, over a boundary annulus in weight space of sufficiently small radius, the eigencurve for a definite quaternion algebra over $\mathbb{Q}$ decomposes as a disjoint union of components which are finite over weight space. On each component, the slope (i.e. the p-adic valuation of the U_p-eigenvalue) varies linearly with the weight, and in particular the slope tends to zero as the weight approaches the boundary of the weight space. A notable consequence of this result is that every irreducible component of the eigencurve contains a classical weight 2 point (with p-part of the Nebentypus character of large conductor, so that the corresponding weight-character is close to the boundary of weight space).

The purpose of this note is to explain that the method of [LWX] can be used to establish a similar result for certain one-dimensional eigenvarieties interpolating Hilbert modular forms (Proposition 2.7.1). More precisely, we assume that the (odd) prime p splits completely in a totally real field F, and consider 'partial eigenvarieties' whose classical points are automorphic forms for totally definite quaternion algebras over F whose weights are only allowed to vary at a single place $v|p$.[1]

We give two applications of this result. The first application, following methods and results of Nekovář (for example, [Nek07]) and Pottharst–Xiao [PX], is to establish new cases of the parity part of the Bloch–Kato conjecture for the Galois representations associated to Hilbert modular forms (see the next subsection for a precise statement). The idea of the proof is that, using the results of *loc. cit.*, we can reduce the parity conjecture for a Hilbert modular newform g of general (even) weight to the parity conjecture for parallel weight two Hilbert modular forms, by moving in a p-adic family connecting g to a parallel weight two form. This parallel weight two form will have local factors at places dividing p given by ramified principal series representations (moving to the boundary of weight space corresponds to increasing the conductor of the ratio of the characters defining this principal series representation). Using our results on the partial eigenvarieties, we carry out this procedure in $d = [F : \mathbb{Q}]$ steps, where each step moves one of the weights to two. An additional difficulty when $F \neq \mathbb{Q}$ is to ensure that at each step we move to a point which is non-critical at all the places $v'|p$, so that the global triangulation over the family provided by the results of [KPX14] specialises to a triangulation at all of these points. At the fixed $v' = v$, this follows automatically from the construction. At $v' \neq v$ we show, roughly speaking, that the locus of points on the partial eigenvariety which are critical at v' forms a union of connected components, so one cannot move from non-critical to critical points.

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[1] One can also consider the slightly more general situation where the assumption is only that the place v is split.

The second application (see Section 4) is to establish that every irreducible component of the 'full' eigenvariety (with dimension $[F:\mathbb{Q}]+1$) for a totally definite quaternion algebra over F contains a classical point with non-critical slopes and parallel weight 2, or any other algebraic weight.

1.2. **The Bloch–Kato conjecture.** Let g be a normalised cuspidal Hilbert modular newform of weight $(k_1, k_2, \ldots, k_d, w = 2)$ and level $\Gamma_0(\mathfrak{n})$, over a totally real number field F, with each k_i even. We suppose moreover that the automorphic representation associated to g has trivial central character.[2]

Let E be the (totally real) number field generated by the Hecke eigenvalues $t_v(g)$. For any finite place λ of E, with residue characteristic p, denote by $V_{g,\lambda}$ the two-dimensional (totally odd, absolutely irreducible) E_λ-representation associated to g which satisfies

$$\det(X - \mathrm{Frob}_v|_{V_{g,\lambda}}) = X^2 - t_v(g)X + q_v$$

for all $v \nmid \mathfrak{n}p$ (Frob_v denotes arithmetic Frobenius).

Note that we have $V_{g,\lambda} \cong V_{g,\lambda}^*(1)$. The conjectures of Bloch and Kato predict the following formula relating the dimension of the Bloch–Kato Selmer group of $V_{g,\lambda}$ to the central order of vanishing of an L-function:

Conjecture 1.2.1.

$$\dim_{E_\lambda} H^1_f(F, V_{g,\lambda}) = \mathrm{ord}_{s=0} L(V_{g,\lambda}, s)\,(\text{or, equivalently, } = \mathrm{ord}_{s=0} L(g,s))\,.$$

In this conjecture, the L-functions are normalised so that the local factors (at good places) are

$$L_v(V_{g,\lambda}, s) = L_v(g,s) = \left(1 - t_v q_v^{-s-1} + q_v^{-2s-1}\right)^{-1}.$$

We refer to the following as the 'parity conjecture for $V_{g,\lambda}$':

Conjecture 1.2.2.

$$\dim_{E_\lambda} H^1_f(F, V_{g,\lambda}) \equiv \mathrm{ord}_{s=0} L(V_{g,\lambda}, s) \pmod 2.$$

Our main result towards the parity conjecture is the following, which is proved in Section 3:

Theorem 1.2.3. *Let p be an odd prime and let F be a totally real number field in which p splits completely. Let g be a cuspidal Hilbert modular newform for F with weight $(k_1, k_2, \ldots, k_d, 2)$ (and each k_i even) such that the associated automorphic representation π has trivial central character. Let E be the number field generated by the Hecke eigenvalues of g and let λ be a finite place of E with residue characteristic p.*

Suppose that for every place $v|p$, π_v is not supercuspidal (in other words, g has finite slope, up to twist, for every $v|p$). If $[F:\mathbb{Q}]$ is odd suppose moreover that there is a finite place $v_0 \nmid p$ of F such that π_{v_0} is not principal series. Then the parity conjecture holds for $V_{g,\lambda}$.

Remark 1.2.4.
(1) When $k_i = 2$ for all i, the above theorem is already contained in [PX]. Moreover, stronger results (without the finite slope hypothesis) are given by [Nek13, Thm. 1.4] and [Nek, Thm. C]. As far as the authors are aware, the only prior results for higher weights are for p-ordinary forms (for example, [Nek06, Thm. 12.2.3]), except when $F = \mathbb{Q}$ in which case the above theorem follows from the results of [PX] and [LWX] (c.f. [LWX, Remark 1.10]), and there is also a result of Nekovář which applies under a mild technical hypothesis [Nek13, Thm. B].
(2) When $[F:\mathbb{Q}]$ is odd we require the existence of the place v_0 in the above theorem in order to switch to a totally definite quaternion algebra.
(3) For fixed g there are infinitely many p satisfying the assumptions of the above theorem. So combining our theorem with the conjectural 'independence of p' for the (parity of the) rank of the Selmer group $H^1_f(F, V_{g,\lambda})$ would imply the parity conjecture for the entire compatible family of Galois representations associated to g.

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[2]Here we use the notation and conventions of [Sai09, §1] for the weights of Hilbert modular forms, and in particular $w = 2$ corresponds to the central character of the associated automorphic representation having trivial algebraic part. Note that in the body of the text this will correspond to taking $w = 0$ in the weights we define for totally definite quaternion algebras.

2. A HALO FOR THE PARTIAL EIGENVARIETY

2.1. **Notation.** We fix an odd prime number p[3], a totally real number field F with $[F:\mathbb{Q}]=d>1$, and we assume that p splits completely in F. We label the places dividing p as $v_1,\dots,v_d$. We also fix a totally definite quaternion algebra D over F, with discriminant δ_D, and assume that p is coprime to δ_D. We fix a maximal order $\mathcal{O}_D\subset D$ and an isomorphism $\mathcal{O}_D\otimes_{\mathcal{O}_F}\widehat{\mathcal{O}}_F^{\delta_D}\cong M_2(\widehat{\mathcal{O}}_F^{\delta_D})=\prod_{v\nmid\delta_D}M_2(\mathcal{O}_{F,v})$. Using this isomorphism, we henceforth identify $\mathcal{O}_{D,p}^\times$ and $\prod_{v|p}\mathrm{GL}_2(\mathcal{O}_{F,v})$. We fix the uniformiser $\varpi_v=p$ of $\mathcal{O}_{F,v}$ for each $v|p$.

For $v|p$, we let $T_{0,v}=\begin{pmatrix}\mathcal{O}_{F,v}^\times & 0\\ 0 & \mathcal{O}_{F,v}^\times\end{pmatrix}\subset\mathrm{GL}_2(\mathcal{O}_{F,v})$ and let

$$I_{v,n}=\left\{\begin{pmatrix}a & b\\ \varpi_v^n c & d\end{pmatrix}\mid a,b,c,d\in\mathcal{O}_{F,v}\right\}\cap\mathrm{GL}_2(\mathcal{O}_{F,v}),$$

$$\overline{I}_{v,n}=\left\{\begin{pmatrix}a & \varpi_v^n b\\ c & d\end{pmatrix}\mid a,b,c,d\in\mathcal{O}_{F,v}\right\}\cap\mathrm{GL}_2(\mathcal{O}_{F,v})$$

and

$$I_{v,1,n}=\left\{\begin{pmatrix}a & b\\ \varpi_v c & d\end{pmatrix}\in I_{v,1}\mid b\equiv 0\pmod{\varpi_v^n}\right\},$$
$$I'_{v,1,n}=\left\{\begin{pmatrix}a & \varpi_v^n b\\ \varpi_v c & d\end{pmatrix}\in I_{v,1,n}\mid a\equiv d\equiv 1\pmod{\varpi_v^{n+1}}\right\}.$$

Set

$$\Sigma_v^+=\left\{\begin{pmatrix}1 & 0\\ 0 & \varpi_v^\beta\end{pmatrix}\mid\beta\in\mathbb{Z}_{\geq 0}\right\}$$

and

$$\Sigma_v^{cpt}=\left\{\begin{pmatrix}1 & 0\\ 0 & \varpi_v^\beta\end{pmatrix}\mid\beta\in\mathbb{Z}_{>0}\right\}.$$

Fix a place $v|p$, and let $K^v=\prod_{v'\neq v}K_{v'}$ be a compact open subgroup of $(\mathcal{O}_D\otimes_F\mathbb{A}_{F,f}^v)^\times$ such that $K=K^vI_{v,1}$ is a *neat* subgroup of $(D\otimes_F\mathbb{A}_{F,f})^\times$. More precisely, we assume that $x^{-1}D^\times x\cap K\subset\mathcal{O}_F^{\times,+}$ for all $x\in(D\otimes_F\mathbb{A}_{F,f})^\times$.

We make the following general definitions:

Definition 2.1.1. Let K be a compact open subgroup of $(D\otimes_F\mathbb{A}_{F,f})^\times$ and let N be a monoid with $K\subset N\subset(D\otimes_F\mathbb{A}_{F,f})^\times$. Suppose M is a left $R[N]$-module (for some commutative coefficient ring R).

(1) If $f:D^\times\backslash(D\otimes_F\mathbb{A}_{F,f})^\times\to M$ is a function and $\gamma\in N$ we define a function ${}_\gamma|f$ by ${}_\gamma|f(g)=\gamma f(g\gamma)$.

(2) We define

$$H^0(K,M)=\{f:D^\times\backslash(D\otimes_F\mathbb{A}_{F,f})^\times\to M\mid {}_u|f=f\text{ for all }u\in K\}.$$

(3) If $K'\subset K$ is a compact open subgroup then we can define a double coset operator

$$[KgK']:H^0(K',M)\to H^0(K,M)$$

for any $g\in N$ by decomposing the double coset $KgK'=\coprod_i g_iK'$ and defining

$$[KgK']f=\sum_i{}_{g_i}|f.$$

The following lemma is easily checked.

[3]The assumption that p is odd is not essential for this section, but it appears in the results of [PX] and [Nek] which we will apply later, so we have excluded the case $p=2$ throughout this paper for simplicity.

Lemma 2.1.2. *Suppose K is a neat compact open subgroup of $(D \otimes_F \mathbb{A}_{F,f})^\times$. Let $g_1, \dots, g_t$ be double coset representatives for $D^\times \backslash (D \otimes_F \mathbb{A}_{F,f})^\times / K$, and let M be an $R[K]$-module on which $Z(K) := \mathcal{O}_F^\times \cap K = \mathcal{O}_F^{\times,+} \cap K$ acts trivially (the equality follows from the neatness assumption).*

Then the R-module map

$$f \mapsto (f(g_1), \dots, f(g_t))$$

gives an isomorphism $H^0(K, M) \cong M^{\oplus t}$.

Now suppose we have an integer w and a tuple of integers

$$k^v = (k_{v'})_{v'|p, v' \neq v}$$

such that $k_{v'} \equiv w \pmod 2$ and $k_{v'} \geq 2$ for all v'.

For each v' we define a finite free $\mathbb{Z}_p = \mathcal{O}_{F_{v'}}$-module with a left action of $K_{v'}$:

$$\mathcal{L}_{v'}(k_{v'}, w) = \det{}^{(w-k_{v'}+2)/2} \operatorname{Sym}^{k_{v'}-2} \mathcal{O}_{F_{v'}}^2$$

where $\mathcal{O}_{F_{v'}}^2$ is the standard representation of $\mathrm{GL}_2(\mathcal{O}_{F,v'}) \cong \mathcal{O}_{D,v'}^\times$.

We then define a finite free $\mathbb{Z}_p$-module with a left action of K^v:

$$\mathcal{L}^v(k^v, w) := \bigotimes_{v'|p, v' \neq v} \mathcal{L}_{v'}(k_{v'}, w)$$

Let χ^v be the character of $T_0^v = \prod_{v'|p, v' \neq v} T_{0,v'}$ given by the highest weight of $\mathcal{L}^v(k^v, w)$ (with respect to the upper triangular Borel). Explicity, this character is

$$\chi^v(t_1, t_2) := \prod_{v'} (t_{1,v'})^{\frac{w+k_{v'}-2}{2}} (t_{2,v'})^{\frac{w-k_{v'}+2}{2}}$$

We now recall some terminology from [JNa]. We let R be a Banach–Tate $\mathbb{Z}_p$-algebra [JNa, 3.1] with a multiplicative pseudo-uniformiser ϖ, and let $\kappa : T_{0,v} \to R^\times$ be a continuous character. Assume that the norm on R is adapted to κ [JNa, Def. 3.3.2] and that $\chi^v \kappa$ is trivial on $Z(K)$. We call such a κ a *weight*.

For $r \geq r_\kappa$ we get a Banach R-module of distributions $\mathcal{D}_\kappa^r$ [JNa, Def. 3.3.9] equipped with a left action of the monoid $\Delta_v = I_{v,1} \Sigma_v^+ I_{v,1}$. This is a space of distributions, and we also have (see the end of [JNa, §3.3]) a space of functions $\mathcal{A}_\kappa^{<r}$ on $\begin{pmatrix} 1 & 0 \\ \varpi_v \mathcal{O}_{F,v} & 1 \end{pmatrix}$ (which we identify with functions in a single variable x on $\varpi_v \mathcal{O}_{F,v}$) with a right action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_v$ given by

$$(f \cdot \gamma)(x) = \kappa(a + bx, (\det \gamma) |\det \gamma|_v / (a + bx)) f\left(\frac{c + dx}{a + bx}\right).$$

There is a natural pairing $\mathcal{D}_\kappa^r \times \mathcal{A}_\kappa^{<r} \to R$ which identifies $\mathcal{A}_\kappa^{<r}$ as the dual of $\mathcal{D}_\kappa^r$ (and not the other way around!), and hence gives an embedding of $\mathcal{D}_\kappa^r$ into the dual of $\mathcal{A}_\kappa^{<r}$. The R-module $H^0(K, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$ is the space of p-adic automorphic forms with fixed weights away from v which we will be studying in this section.

If R is a $\mathbb{Q}_p$-algebra, we have a natural action of $(D \otimes_F \mathbb{A}_{F,f}^v)^\times \times \Delta_v$ on $\mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r$, and we get associated double coset operators.

We will especially consider the action on $H^0(K, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$ of the Hecke operator $U_v = [K \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K]$. This Hecke operator is compact because, by [JNa, Cor. 3.3.10], it factors as a composition

$$H^0(K, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r) \to H^0(K, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^{r^{1/p}}) \hookrightarrow H^0(K, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$$

where the second map is induced by the natural compact inclusion $\mathcal{D}_\kappa^{r^{1/p}} \hookrightarrow \mathcal{D}_\kappa^r$.

We will also need the (non-compact) Hecke operators $U_{v'} = p^{-(w-k_{v'}+2)/2} [K \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{v'} \end{smallmatrix}\right) K]$ for the places $v \neq v'|p$. We have normalised the operators $U_{v'}$ so that they are consistent with U_v. In particular, when R is a finite field extension of $\mathbb{Q}_p$, the non-zero eigenvalues for $U_{v'}$ will have p-adic valuation between 0 and $k_{v'} - 1$.

2.2. **Locally algebraic weights.** Suppose $\kappa : T_{0,v} \to E^\times$ is a weight, where E is a finite extension of $\mathbb{Q}_p$, and for some positive integer k with $k \equiv w \pmod 2$ the restriction of κ to an open subgroup $(1+\varpi_v^n\mathcal{O}_{F_v})^2$ of $T_{0,v}$ coincides with the character

$$\chi_k(t_1,t_2) = (t_1)^{\frac{w+k-2}{2}}(t_2)^{\frac{w-k+2}{2}}.$$

Then, denoting the finite order character κ/χ_k by ϵ, we say that κ is *locally algebraic of weight* (k,w) *and character* ϵ. If we make the standard identification of the E-dual (right) representation $\mathcal{L}_v(k,w)^\vee$ with homogeneous degree k polynomials in two variables (X,Y) then evaluation at $(1,x)$ gives an injective $I'_{v,1,n-1}$-equivariant map

$$\mathcal{L}_v(k,w)^\vee \to \mathcal{A}_\kappa^{<r}.$$

The action of $\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi_v\end{smallmatrix}\right)$ on the right hand side induces the action of $p^{-(w-k+2)/2}\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi_v\end{smallmatrix}\right)$ on the left hand side. Taking duals and using the embedding $\mathcal{D}_\kappa^r \hookrightarrow (\mathcal{A}_\kappa^{<r})^\vee$, we obtain a surjective $I'_{v,1,n-1}$-equivariant map

$$\mathcal{D}_\kappa^r \to \mathcal{L}_v(k,w)\otimes_{\mathbb{Z}_p} E.$$

Moreover, if we let $\mathcal{L}_v(k,w,\epsilon)$ denote the $I_{v,1,n-1}$-representation obtained from $\mathcal{L}_v(k,w)\otimes_{\mathbb{Z}_p} E$ by twisting the action by ϵ then we obtain a surjective $I_{v,1,n-1}$-equivariant map

$$\mathcal{D}_\kappa^r \to \mathcal{L}_v(k,w,\epsilon).$$

2.3. **Comparing level $I_{v,1}$ and $I_{v,1,n-1}$.** A version of the following lemma is commonly used in Hida theory (see for example [Ger09, Lem. 2.5.2]):

Lemma 2.3.1.

(1) *For $n \geq 2$, the endomorphism U_v of $H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)$ factors through the natural inclusion*

$$H^0(K^vI_{v,1,n-2},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r) \hookrightarrow H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r).$$

(2) *The natural inclusion*

$$\iota : H^0(K^vI_{v,1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r) \hookrightarrow H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)$$

is U_v-equivariant and if $h\in\mathbb{Q}_{\geq 0}$ it induces an isomorphism between U_v-slope $\leq h$ subspaces:

$$\iota : H^0(K^vI_{v,1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)^{\leq h} \cong H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)^{\leq h}$$

Proof. First we note that (for $n\geq 1$) the action of U_v on $H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)$ is given by

$$f \mapsto \sum_{i=0}^{p-1}\left(\begin{smallmatrix}1 & 0\\ \varpi_v i & \varpi_v\end{smallmatrix}\right)|f.$$

The U_v-equivariance in the second statement follows immediately from this. The rest of the second statement follows from the first, since U_v acts invertibly on the slope $\leq h$ subspace.

It follows from a simple computation that the double coset operator $[I_{v,1,n-1}\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi_v\end{smallmatrix}\right)I_{v,1,n-1}]$ is equal to $[I_{v,1,n-2}\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi_v\end{smallmatrix}\right)I_{v,1,n-1}]$, and the first part follows immediately from this. □

Having done all this, composing ι with the map induced by $\mathcal{D}_\kappa^r \to \mathcal{L}_v(k,w,\epsilon)$, we obtain a Hecke-equivariant map

$$\pi : H^0(K^vI_{v,1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r) \to H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{L}_v(k,w,\epsilon)), \tag{2.3.1}$$

where the action of U_v on the target is the $\star$*-action* defined by multiplying the standard action of U_v by $p^{-(w-k+2)/2}$.

Proposition 2.3.2. *Let $h\in\mathbb{Q}_{\geq 0}$ with $h<k-1$. The map π induces an isomorphism between U_v-slope $\leq h$ subspaces.*

Proof. By Lemma 2.3.1 it suffices to show that the map

$$H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r) \to H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{L}_v(k,w,\epsilon))$$

induces an isomorphism between U_v-slope $\leq h$ subspaces, and this can be proved as in [Han, Thm. 3.2.5]. □

2.4. **The 'Atkin–Lehner trick'.** In this section we establish a result analogous to [LWX, Prop. 3.22]. Let $n \geq 2$ be an integer and suppose

$$K_{v'} \subset \begin{pmatrix} 1+\varpi_{v'}^n \mathcal{O}_{F,v'} & \mathcal{O}_{F,v'} \\ \varpi_{v'}^n \mathcal{O}_{F,v'} & 1+\varpi_{v'}^n \mathcal{O}_{F,v'} \end{pmatrix}$$

for each $v'|p$, $v' \neq v$. Note that, combined with our neatness assumption, this implies that $Z(K)$ is contained in $1+p^n\mathcal{O}_F$.

Fix an integer $k \geq 2$ with the same parity as w, together with a finite order character $\epsilon = (\epsilon_1, \epsilon_2) : T_{0,v} \to E^\times$, where ϵ_1 and ϵ_2 are characters of $(\mathcal{O}_{F,v}/\varpi_v^n)^\times$ and ϵ_2/ϵ_1 has conductor (ϖ_v^n).

Let $\epsilon_{\mathbb{Q}}$ be the finite order Hecke character of $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$ associated to the Dirichlet character

$$(\mathbb{Z}/p^n\mathbb{Z})^\times \cong (\mathcal{O}_{F,v}/\varpi_v^n)^\times \xrightarrow{\epsilon_1\epsilon_2} E^\times.$$

We now consider the space of classical automorphic forms $S(k,w,\epsilon) := H^0(K^v I_{v,1,n-1}, \mathcal{L}^v(k^v,w) \otimes \mathcal{L}(k,w,\epsilon))$. By Lemma 2.1.2, the dimension of this space is equal to $(k-1)p^{n-1}t$ (as we noted above, $Z(K) \subset 1+p^n\mathcal{O}_F$, so it acts trivially on the coefficients), where

$$t = |D^\times \backslash (D \otimes_F \mathbb{A}_{F,f})^\times / K^v I_{v,1}| \prod_{v'|p, v' \neq v} (k_{v'}-1).$$

Denote the slopes of U_v appearing in $S(k,w,\epsilon)$ by $\alpha_0(\epsilon), \ldots, \alpha_{(k-1)p^{n-1}t-1}(\epsilon)$ in non-decreasing order.

Lemma 2.4.1. *We have $\alpha_i(\epsilon) = k-1-\alpha_{(k-1)p^{n-1}t-1-i}(\epsilon^{-1})$ for $i = 0, \ldots, (k-1)pt-1$. In particular, the sum (with multiplicities) of the U_v slopes appearing in $S(k,w,\epsilon) \oplus S(k,w,\epsilon^{-1})$ is $(k-1)^2p^{n-1}t$.*

Proof. First we fix an embedding $\iota : E \hookrightarrow \mathbb{C}$. Using this embedding, we regard $\epsilon_{\mathbb{Q}}$ and ϵ as complex valued characters. The space $S(k,w,\epsilon) \otimes_{E,\iota} \mathbb{C}$ can be described in terms of automorphic representations for D. If π contributes to this space, the local factor π_v is a principal series representation of $\mathrm{GL}_2(F_v)$ obtained as the normalised induction of a pair of characters

$$\mathrm{unr}(\alpha^{-1}\zeta p^w)\epsilon_1^{-1} \times \mathrm{unr}(\alpha)\epsilon_2^{-1}$$

where ζ is a root of unity. To see this, we first note that π_v has a non-zero subspace on which $I_{v,1,n-1}$ acts via the character ϵ^{-1} (these are the vectors in π_v which contribute to $S(k,w,\epsilon) \otimes_{E,\iota} \mathbb{C}$). So $\pi_v \otimes \epsilon_2 \circ \det$ has a non-zero subspace on which $I_{v,1,n-1}$ acts via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (\epsilon_2/\epsilon_1)(a)$. Applying $\begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{n-1} \end{pmatrix}$ to this subspace, we get a non-zero subspace where the action of $I_{v,n}$ is given by the same formula. Therefore the conductors of both $\pi_v \otimes \epsilon_2 \circ \det$ and its central character ϵ_2/ϵ_1 are (ϖ_v^n). It follows (e.g. by [Tem14, Lem. 3.3]) that $\pi_v \otimes \epsilon_2 \circ \det$ is the normalised induction of $\mu_1 \times \mu_2$ with $\mu_1|_{\mathcal{O}_{F,v}^\times} = \epsilon_2/\epsilon_1$ and $\mu_2|_{\mathcal{O}_{F,v}^\times} = 1$. The rest of the claim is deduced from the fact that the central character of π_v is the product of a finite order unramified character and $\epsilon_1^{-1}\epsilon_2^{-1}|\cdot|_v^{-w}$.

We now need to compute the eigenvalue for the standard U_v action on the subspace of π_v where $I_{v,1,n-1}$ acts via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \epsilon_1(a)^{-1}\epsilon_2(d)^{-1}$. Conjugating by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$, we can do the same for the standard U_v action defined with respect to $\overline{I}_{v,n}$, and we get U_v-eigenvalue $\alpha p^{1/2}$. Using the $\star$-action we therefore get U_v-eigenvalue $\alpha p^{\frac{k-1-w}{2}}$.

Twisting by $\epsilon_{\mathbb{Q}}$ gives a π' with local factor π'_v the normalised induction of

$$\mathrm{unr}(\alpha^{-1}\zeta p^w)\epsilon_2 \times \mathrm{unr}(\alpha)\epsilon_1,$$

which contributes to $S(k,w,\epsilon^{-1})$ with $\star$-action U_v-eigenvalue $\alpha^{-1}\zeta p^{\frac{w+k-1}{2}}$. Summing the two slopes together we get $(k-1)$, which gives the desired result. □

2.5. **The weight space of the partial eigenvariety.** We are going to construct a 'partial eigenvariety' out of the spaces $H^0(K, \mathcal{L}^v(k^v,w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$. The underlying weight space $\mathcal{W}$ is defined by letting

$$\mathcal{W}(A) = \{\kappa \in \mathrm{Hom}_{cts}(T_{0,v}, A^\times) : \chi^v\kappa|_{Z(K)} = 1\}$$

for algebras A of topologically finite type over $\mathbb{Q}_p$. We let Δ denote the torsion subgroup of $\mathcal{O}_{F,v}^\times$.

Lemma 2.5.1. *We have an isomorphism $\mathcal{W}_{\mathbb{C}_p} \cong \coprod_{\eta,\omega} D_{\mathbb{C}_p}$ where D is the open unit disc and η, ω run over pairs of characters $\eta : \mathcal{O}_v^\times \to \mathbb{C}_p^\times, \omega : \Delta \times \Delta \to \mathbb{C}_p^\times$ such that $\chi^v \eta|_{Z(K)} = 1$ and ω restricted to the diagonal copy of Δ is equal to $\eta|_\Delta$.*

The isomorphism is given by taking κ to $\kappa(\exp(p), \exp(p)^{-1}) - 1$ in the component labelled by $\eta(t_v) = \kappa(t_v, t_v)$ and $\omega(\delta_1, \delta_2) = \kappa(\delta_1, \delta_2)$.

Proof. This follows from the fact that the closure of $Z(K)$ in $T_{0,v}$ is a finite index subgroup of $\mathcal{O}_v^\times$ centrally embedded in $T_{0,v}$. □

Remark 2.5.2. The condition that $\chi^v \eta|_{Z(K)} = 1$ is equivalent to the condition that $\eta(t_v) = t_v^w \tilde{\eta}(t_v)$ where $\tilde{\eta}$ is a finite order character, trivial on $Z(K)$. In particular, there are finitely many possibilities for η. Moreover, for each η, if we denote by E_η the finite extension of $\mathbb{Q}_p$ generated by the values of η then the open and closed immersion $\coprod_\omega D_{\mathbb{C}_p} \hookrightarrow \mathcal{W}_{\mathbb{C}_p}$ given by restricting to the components labelled by η is defined over E_η. For each pair η, ω we therefore denote by $\mathcal{W}_{\eta,\omega}$ the corresponding connected component (which is isomorphic to the open unit disc over E_η) of $\mathcal{W}_{E_\eta}$.

If κ is a point of $\mathcal{W}_{\mathbb{C}_p}$ we denote by z_κ the corresponding point of $D_{\mathbb{C}_p}$. If $r \in (0,1) \cap \mathbb{Q}$, the union of open annuli given by $r < |z_\kappa|_p < 1$ is denoted by $\mathcal{W}^{>r}$.

2.6. **Lower bound for the Newton polygon.** In this section we establish a result analogous to [LWX, Thm. 3.16], following the approach of [JNa, §6.2]. We now return to our general situation: R is a Banach–Tate $\mathbb{Z}_p$-algebra with a multiplicative pseudo-uniformiser ϖ, and $\kappa : T_{0,v} \to R^\times$ is a continuous character such that the norm on R is adapted to κ and $\chi^v \kappa$ is trivial on $Z(K)$.

As in [JNa, (3.2.1)], for $\alpha \in \mathbb{Z}_{\geq 0}$, we define

$$n(r, \varpi, \alpha) = \left\lfloor \frac{\alpha \log_p r}{\log_p |\varpi|} \right\rfloor.$$

Lemma 2.6.1. *Assume that there is no $x \in R$ with $1 < |x| < |\varpi|^{-1}$. Let*

$$t = |D^\times \backslash (D \otimes_F \mathbb{A}_{F,f})^\times / K^v I_{v,1}| \prod_{v' | p, v' \neq v} (k_{v'} - 1)$$

If we define

$$\lambda(0) = 0, \lambda(i+1) = \lambda(i) + n(r, \varpi, \lfloor i/t \rfloor) - n(r^{1/p}, \varpi, \lfloor i/t \rfloor)$$

then the Fredholm series

$$\det(1 - TU_v | H^0(K^v I_{v,1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)) = \sum_{n \geq 0} c_n T^n \in R\{\{T\}\}$$

satisfies $|c_n| \leq |\varpi|^{\lambda(n)}$.

Proof. This is essentially [JNa, Lemma 6.2.1]. Our global set-up is slightly different to that in *loc. cit.*, but the proof goes through verbatim. □

We now fix a component $\mathcal{W}_{\eta,\omega}$ of weight space. Let (ϖ^n) be the conductor of $\tilde{\eta}$ so we have $E := E_\eta = \mathbb{Q}_p(\zeta_{p^{n-1}}) \subset \mathbb{C}_p$ (in particular, $E = \mathbb{Q}_p$ if $\tilde{\eta}$ is trivial). We fix a uniformiser $\varpi_E \in E$ and normalise the absolute value $|\cdot|_E$ on E by $|p| = p^{-1}$. Let $\Lambda = \mathcal{O}_E[[X]]$. We have a universal weight

$$\kappa_{(\eta,\omega)} : T_{0,v} \to \Lambda^\times$$

determined by $\kappa|_{\Delta \times \Delta} = \omega$, $\kappa(t_v, t_v) = \eta(t_v)$ and $\kappa(\exp(p), \exp(p)^{-1}) = 1 + X$.

We give the complete local ring Λ the $\mathfrak{m}_\Lambda = (\varpi_E, X)$-adic topology.

Let $\mathfrak{W}_{\eta,\omega} = \mathrm{Spa}(\Lambda, \Lambda)$, denote its analytic locus by $\mathcal{W}_{\eta,\omega}^{an}$ and let $\mathcal{U}_1 \subset \mathcal{W}_{\eta,\omega}^{an}$ be the rational subdomain defined by

$$\mathcal{U}_1 = \{|\varpi_E| \leq |X| \neq 0\}.$$

Pulling back $\mathcal{U}_1$ to the rigid analytic open unit disc $\mathcal{W}_{\eta,\omega}$ gives the 'boundary annulus' $|X| \geq |\varpi_E|_E^{-1}$.

We let $R = \mathcal{O}(\mathcal{U}_1)$. More explicitly, we can describe the elements of R° as formal power series

$$\left\{ \sum_{n \in \mathbb{Z}} a_n X^n : a_n \in \mathcal{O}_E, |a_n \varpi_E^n|_E \leq 1, |a_n \varpi_E^n|_E \to 0 \text{ as } n \to -\infty \right\}.$$

X is a topologically nilpotent unit in R and so equipping R with the norm $|r| = \inf\{|\varpi_E|_E^n \mid r \in X^n R^\circ, n \in \mathbb{Z}\}$ makes R into a Banach–Tate $\mathbb{Z}_p$-algebra. This norm has the explicit description:

$$|\sum_{n\in\mathbb{Z}} a_n X^n| = \sup\{|a_n \varpi_E^n|_E\}. \tag{2.6.1}$$

Note that X is a multiplicative pseudo-uniformiser and there is no $x \in R$ with $1 < |x| < |X|^{-1}$.

Lemma 2.6.2. *The norm we have defined on R is adapted to $\kappa_{(\eta,\omega)}$. Moreover, for $t \in T_1 = (1+\varpi_v\mathcal{O}_{F,v})^2$ we have $|\kappa_{(\eta,\omega)}(t) - 1| \leq |\varpi_E|_E$.*

Proof. If $t \in T_1$ we have $\kappa_{(\eta,\omega)}(t) - 1 = u\zeta(1+X)^\alpha - 1$ for some $\alpha \in \mathbb{Z}_p$, $u \in 1 + p\mathcal{O}_E$ and $\zeta \in \mathcal{O}_E$ with $\zeta^{p^{n-1}} = 1$. So $|\kappa_{(\eta,\omega)}(t) - 1| \leq |\varpi_E|_E$. □

We can now apply Lemma 2.6.1.

Corollary 2.6.3. *Consider the Fredholm series*

$$\det(1 - TU_v | H^0(K^v I_{v,1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^{|\varpi_E|_E})) = \sum_{n\geq 0} c_n T^n \in R\{\{T\}\}.$$

Let

$$t = |D^\times \backslash (D \otimes_F \mathbb{A}_{F,f})^\times / K^v I_{v,1}| \prod_{v'|p, v'\neq v} (k_{v'} - 1).$$

(1) *We have $c_n = \sum_{m\geq 0} b_{n,m} X^m \in \Lambda$ for all n.*
(2) *Moreover, $|b_{n,m}\varpi_E^m|_E \leq |\varpi_E^{\lambda(n)}|_E$ for all $m, n \geq 0$, where $\lambda(0) = 0, \lambda(1), \ldots$ is a sequence of integers determined by*

$$\lambda(0) = 0, \lambda(i+1) = \lambda(i) + \lfloor i/t \rfloor - \lfloor i/pt \rfloor.$$

(3) *For $z \in \mathbb{C}_p$ with $0 < v_p(z) < v_p(\varpi_E)$, we have $v_p(c_n(z)) \geq \lambda(n) v_p(z)$ for every $n \geq 0$, with equality holding if and only if $b_{n,\lambda(n)} \in \mathcal{O}_E^\times$. If $b_{n,\lambda(n)} \notin \mathcal{O}_E^\times$, then*

$$v_p(c_n(z)) \geq \lambda(n) v_p(z) + \min\{v_p(z), v_p(\varpi_E) - v_p(z)\}.$$

Proof. The first two parts follow from Lemma 2.6.1, exactly as in [JNa, Thm. 6.3.2]. Note that $n(|\varpi_E|_E, X, \lfloor i/t \rfloor) = \lfloor i/t \rfloor$ and $n(|\varpi_E|_E^{1/p}, X, \lfloor i/t \rfloor) = \lfloor 1/p \lfloor i/t \rfloor \rfloor = \lfloor i/pt \rfloor$.

Let $z \in \mathbb{C}_p$ with $0 < v_p(z) < v_p(\varpi_E)$. It follows immediately from the second part that $v_p(c_n(z)) \geq \lambda(n) v_p(z)$ for every $n \geq 0$, with equality holding if and only if $b_{n,\lambda(n)} \in \mathcal{O}_E^\times$. Finally, if $b_{n,\lambda(n)} \notin \mathcal{O}_E^\times$ then $v_p(b_{n,\lambda(n)}) \geq v_p(\varpi_E)$ and the rest of the third part is easy to check. □

2.7. **The spectral curve and partial eigenvariety.** We can now construct the spectral curve $\mathcal{Z}^{U_v}(k^v, w) \to \mathcal{W}$ for the compact operator U_v acting on the spaces $H^0(K^v I_{v,1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$, for $\kappa = \kappa_U$ ranging over the weights induced from affinoid open $U \subseteq \mathcal{W}$, as well as the pullback $\mathcal{Z}^{U_v}(k^v, w)_{\eta,\omega}^{>r} \to \mathcal{W}_{\eta,\omega}^{>r}$ for $r \in (0,1) \cap \mathbb{Q}$ and a component $\mathcal{W}_{\eta,\omega}$ of weight space.

Proposition 2.7.1. *Fix a component $\mathcal{W}_{\eta,\omega}$ of weight space, and let $E = E_\eta = \mathbb{Q}_p(\zeta_{p^m}) \subset \mathbb{C}_p$ be the subfield generated by the image of η as before. Then $\mathcal{Z}^{U_v}(k^v, w)_{\eta,\omega}^{>|\varpi_E|} = \mathcal{Z}_0 \coprod \mathcal{Z}_{(0,1)} \coprod \mathcal{Z}_1 \coprod \mathcal{Z}_{(1,2)} \coprod \cdots$ is a disjoint union of rigid analytic spaces $\mathcal{Z}_I$ (with I denoting an interval $[i,i]$ or $(i, i+1)$) which are finite and flat over $\mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$. For each point $x \in \mathcal{Z}_I$, with corresponding U_v-eigenvalue λ_x, we have $v_p(\lambda_x) \in (p-1)p^m v_p(z_{\kappa(x)}) \cdot I$.*

Proof. This follows from Corollary 2.6.3 and Lemma 2.4.1, as in [LWX, Proof of Thm. 1.3]. We sketch the argument. To begin with we set $n = m + 2$. Note that $\tilde{\eta}$ factors through $(\mathcal{O}_{F,v}/\varpi^{n-1})^\times$. By passing to a normal compact open subgroup of K^v we may assume that

$$K_{v'} \subset \begin{pmatrix} 1 + \varpi_{v'}^n \mathcal{O}_{F,v'} & \mathcal{O}_{F,v'} \\ \varpi_{v'}^n \mathcal{O}_{F,v'} & 1 + \varpi_{v'}^n \mathcal{O}_{F,v'} \end{pmatrix}$$

for each $v'|p$, $v' \neq v$. Now we consider points $\kappa \in \mathcal{W}(\mathbb{C}_p)$ which are locally algebraic of weight (k_v, w) and character ϵ, with the ϵ_i factoring through $(\mathcal{O}_{F,v}/\varpi^n)^\times$ and ϵ_2/ϵ_1 of conductor (ϖ^n). We furthermore

insist that κ is in the component $\mathcal{W}_{\eta,\omega}$. This amounts to requiring that $\kappa|_{\Delta\times\Delta} = \omega$ and $\tilde{\eta} = \epsilon_1\epsilon_2$. We have $v_p(z_\kappa) = 1/\phi(p^{n-1}) < v_p(\varpi_E)$ so $\kappa \in \mathcal{W}_{\eta,\omega}^{>|\varpi_E|}(\mathbb{C}_p)$.

For $k \in \mathbb{Z}_{\geq 2}$ with $k \equiv w \pmod 2$ we set $n_k = (k-1)p^{n-1}t = \dim S(k,w,\epsilon)$. We now carry out 'Step I' of [LWX, Proof of Thm. 1.3]: we have

$$\begin{aligned} v_p(z_\kappa)\lambda(n_k) &= \frac{1}{\phi(p^{n-1})} \sum_{i=0}^{(k-1)p^{n-1}t-1} \left(\left\lfloor \frac{i}{t} \right\rfloor - \left\lfloor \frac{i}{pt} \right\rfloor \right) \\ &= \frac{1}{\phi(p^{n-1})} \left(t \sum_{i=0}^{(k-1)p^{n-1}-1} i - pt \sum_{i=0}^{(k-1)p^{n-2}-1} i \right) \\ &= \frac{(k-1)^2 p^{n-1} t}{2}. \end{aligned}$$

So, by Lemma 2.4.1, $v_p(z_\kappa)\lambda(n_k)$ is equal to half the sum of the U_v-slopes on $S(k,w,\epsilon)$ and $S(k,w,\epsilon^{-1})$. Combining this with Corollary 2.6.3(3) (at the weights corresponding to both (k,w,ϵ) and (k,w,ϵ^{-1})), we deduce that the sum of the first n_k U_v-slopes on $H^0(K^v I_{v,1}, \mathcal{L}^v(k^v,w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^{|\varpi_E|_E})$ is $\frac{(k-1)^2 p^{n-1} t}{2}$ and that the Newton polygon of $\sum_{n\geq 0} c_n(z_\kappa)T^n$ passes through the point $(n_k, \lambda(n_k)v_p(z_\kappa))$.

'Step II' of [LWX, Proof of Thm. 1.3] can now be carried over verbatim (replacing q with p^{n-1}), and this establishes the rest of the proposition. □

Remark 2.7.2. In the above Proposition, we are proving that the partial eigenvariety exhibits 'halo' behaviour over a boundary annulus whose radius depends on the conductor of the character $\tilde{\eta}$. We do not know to what extent this dependence is an artifact of our proof.

Now we fix an integer $n \geq 1$ and assume that $K_{v'} = I'_{v',1,n-1}$ for each place $v'|p$ with $v' \neq v$. We set $K = K^v I_{v,1}$. Let S be the set of finite places w of F where either $w|p$, D_w is non-split or D_w is split but $K_w \neq \mathcal{O}_{D,w}^\times$. For $w \notin S$ we have Hecke operators

$$S_w = [K \begin{pmatrix} \varpi_w & 0 \\ 0 & \varpi_w \end{pmatrix} K],\ T_w = [K \begin{pmatrix} 1 & 0 \\ 0 & \varpi_w \end{pmatrix} K]$$

which are independent of the choice of uniformiser $\varpi_w \in F_w$.

The spaces $H^0(K, \mathcal{L}^v(k^v,w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$ give rise to a coherent sheaf $\mathcal{H}$ over $\mathcal{Z}^{U_v}(k^v,w)$, equipped with an action of the Hecke operators $\{U_{v'}, v'|p\}$ and $\{S_w, T_w : w \notin S\}$. If we let $\mathbb{T}$ denote the free commutative $\mathbb{Z}_p$-algebra generated by these Hecke operators, and let $\psi : \mathbb{T} \to \mathrm{End}(\mathcal{H})$ be the map induced by the Hecke action, then we have an eigenvariety datum $(\mathcal{W}, \mathcal{Z}^{U_v}(k^v,w), \mathcal{H}, \mathbb{T}, \psi)$ (see for example [Han, §4.2], [JNb, §3.1] for the notion of eigenvariety data and more details on the eigenvariety construction). We denote the associated eigenvariety by $\mathcal{E}(k^v,w)$ and, if $r \in (0,1)$, denote its pullback to $\mathcal{W}^{>r}$ by $\mathcal{E}(k^v,w)^{>r}$.

Definition 2.7.3. A *classical point* of $\mathcal{E}(k^v,w)$ is a point with locally algebraic weight corresponding to a Hecke eigenvector with non-zero image under the map π of (2.3.1), whose systems of Hecke eigenvalue do not arise from one-dimensional automorphic representations of $(D \otimes_F \mathbb{A}_F)^\times$.

The points excluded in the above definition do not correspond to classical Hilbert modular forms under the Jacquet–Langlands correspondence. They all have parallel weight 2, and their $U_{v'}$ eigenvalues $\alpha_{v'}$ satisfy $v_p(\alpha_{v'}) = 1$ for all $v'|p$.

Proposition 2.7.4.

(1) *$\mathcal{E}(k^v,w)$ is equidimensional of dimension 1 and flat over $\mathcal{W}$.*
(2) *Let C be an irreducible component of $\mathcal{E}(k^v,w)$. There exists a positive integer c (depending only on C) such that if $n \geq c$ and ζ is a p^nth root of unity and $k \geq 2$ is an integer with the same parity as w then C contains a point which is locally algebraic of weight (k,w) and U_v-slope $< k-1$ (in particular, this point is classical).*

Proof. (1) This part is proved in the same way as the well-known analogous statement for the Coleman–Mazur eigencurve (for example, see [JNa, §6.1]).

(2) This is a consequence of Proposition 2.7.1. Indeed, C maps to a single component $\mathcal{W}_{\eta,\omega}$ of weight space, and we let E be as in that proposition. Then any irreducible component C' of $C^{>|\varpi_E|}$ is finite flat over an irreducible component of $\mathcal{Z}^{U_v}(k^v,w)^{>|\varpi_E|}$. It follows from Proposition 2.7.1 that C' is finite flat over $\mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$ and there is an interval $I=[i,i]$ or $(i,i+1)$ such that for each point $x\in C'$, with corresponding U_v-eigenvalue λ_x, we have $v_p(\lambda_x)\in\phi(p^{m+1})v_p(z_{\kappa(x)})\cdot I$. If $v_p(z_{\kappa(x)})$ is sufficiently small, then $v_p(\lambda_x)<k-1$. To finish the proof, we note that we can find a locally algebraic point of weight (k,w) in $\mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$ with $v_p(z_{\kappa(x)})$ as small as we like, by choosing a character $\epsilon=(\epsilon_1,\epsilon_2)$ (compatible with η and ω) such that $\epsilon_1(\exp(p))/\epsilon_2(\exp(p))$ is a p-power root of unity of sufficiently large order.

□

3. The parity conjecture

We can now apply the preceding results to establish some new cases of the parity conjecture for Hilbert modular forms, using [PX, Thms. A and B]. First we need to discuss the family of Galois representations carried by the partial eigenvarieties and their p-adic Hodge theoretic properties.

3.1. Galois representations.

We begin by noting that there is a continuous 2-dimensional pseudocharacter $T:G_F\to\mathcal{O}_{\mathcal{E}(k^v,w)}$ with $T(\mathrm{Frob}_w)=T_w$ for all $w\notin S$. This follows from [Che04, Prop. 7.1.1], the Zariski density of classical points in $\mathcal{E}(k^v,w)$ and the existence of Galois representations associated to Hilbert modular forms.

We denote the normalization of $\mathcal{E}(k^v,w)$ by $\tilde{\mathcal{E}}$, and we also denote by T the pseudocharacter on $\tilde{\mathcal{E}}$ obtained by pullback from $\mathcal{E}(k^v,w)$. We denote by $\tilde{\mathcal{E}}^{\mathrm{irr}}\subset\tilde{\mathcal{E}}$ the (Zariski open, see [Che14, Example 2.20]) locus where T is absolutely irreducible. Note also that $\tilde{\mathcal{E}}^{\mathrm{irr}}$ is Zariski dense in $\tilde{\mathcal{E}}$, since classical points have irreducible Galois representations.

Proposition 3.1.1. *There is a locally free rank 2 $\mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}$-module V equipped with a continuous representation*

$$\rho:G_{F,S}\to\mathrm{GL}_{\mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}}(V)$$

satisfying

$$\det(X-\rho(\mathrm{Frob}_w))=X^2-T_wX+q_wS_w$$

for all $w\notin S$ (where Frob_w denotes an arithmetic Frobenius).

Proof. First we recall that T canonically lifts to a continuous representation $\rho:G_{F,S}\to\mathcal{A}^\times$ where $\mathcal{A}$ is an Azumaya algebra of rank 4 over $\tilde{\mathcal{E}}^{\mathrm{irr}}$, by [Che04, Cor. 7.2.6] (see also [Che14, Prop. G]). It remains to show that $\mathcal{A}$ is isomorphic to the endomorphism algebra of a vector bundle.

On the other hand, [CM98, Thm. 5.1.2] and the remark appearing after this theorem shows that there is an admissible cover of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ by affinoid opens $\{U_i\}_{i\in I}$, together with continuous representations

$$\rho_i:G_{F,S}\to\mathrm{GL}_2(\mathcal{O}(U_i))$$

lifting the pseudocharacters $T|_{U_i}$.

By the uniqueness part of [Che04, Lem. 7.2.4], we deduce that for each affinoid open U_i, we have an isomorphism of $\mathcal{O}(U_i)$-algebras $\mathcal{A}(U_i)\cong M_2(\mathcal{O}(U_i))$. Now, by the standard argument relating the Brauer group to cohomology (see [Mil80, Thm. IV.2.5]; the constructions using either Čech cohomology or gerbes can be applied, since Čech cohomology coincides with usual cohomology on $\tilde{\mathcal{E}}^{\mathrm{irr}}$ by [vdP82, Prop. 1.4.4]) we can associate to $\mathcal{A}$ an element $F_{\mathcal{A}}$ of $H^2(\tilde{\mathcal{E}}^{\mathrm{irr}},\mathcal{O}^\times_{\tilde{\mathcal{E}}^{\mathrm{irr}}})$ (crucially, this is the cohomology on the rigid analytic site, not étale cohomology).

This element is trivial if and only if $\mathcal{A}$ is isomorphic to the endomorphism algebra of a vector bundle.

Since $\tilde{\mathcal{E}}^{\mathrm{irr}}$ is separated and one-dimensional, $H^2(\tilde{\mathcal{E}}^{\mathrm{irr}},\mathcal{F})$ vanishes for any Abelian sheaf $\mathcal{F}$ by [dJvdP96, Cor. 2.5.10, Rem. 2.5.11]. In particular, $F_{\mathcal{A}}$ is trivial and we are done. □

Remark 3.1.2. In fact, [CM98, Thm. 5.1.2] applies over the whole of $\tilde{\mathcal{E}}$, not just the irreducible locus, so there are representations $\rho_i:\mathcal{O}(U_i)[G_{F,S}]\to M_2(\mathcal{O}(U_i))$ lifting T over each member of an admissible affinoid cover $\{U_i\}_{i\in I}$ of $\tilde{\mathcal{E}}$. Shrinking the cover if necessary, we may assume that the intersection of

any two distinct covering affinoids $U_i \cap U_j$ is contained in $\tilde{\mathcal{E}}^{\mathrm{irr}}$, so we obtain a canonical isomorphism of $\mathcal{O}_{U_i \cap U_j}$-algebras $M_2(\mathcal{O}_{U_i \cap U_j}) \cong M_2(\mathcal{O}_{U_i \cap U_j})$ intertwining the representations ρ_i and ρ_j.

This gives the gluing data necessary to define a Galois representation to an Azumaya algebra, which as above is isomorphic to the endomorphism algebra of a vector bundle (by the vanishing of $H^2(\tilde{\mathcal{E}}, \mathcal{O}_{\tilde{\mathcal{E}}}^\times)$). So we finally obtain a (possibly non-canonical) Galois representation on a vector bundle over $\tilde{\mathcal{E}}$, although in what follows we will just work over the irreducible locus as this suffices for our purposes.

If z is a point of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ we denote by V_z the specialisation of V at z, which we regard as a 2-dimensional representation of G_F over the residue field $k(z)$. Note that we obtain a continuous character $\det \rho : G_F \to \Gamma(\tilde{\mathcal{E}}^{\mathrm{irr}}, \mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}})^\times$ with $\det \rho(z) = \det V_z$. In fact $\det \rho$ is constant on connected components of $\tilde{\mathcal{E}}^{\mathrm{irr}}$. Indeed for every classical point z, $\det \rho(z)$ is Hodge–Tate of weight $-1-w$ at every place dividing p, and the results cited in the proof of Proposition 3.1.8 below show that this property extends to all points z. Class field theory and the fact that Hodge–Tate characters are locally algebraic [Ser98, Thm. 3, Appendix to Ch. III] shows that each $\det \rho(z)$ is the product of a finite order character and the $(1+w)$-power of the cyclotomic character, so $\det \rho$ is constant on connected components.

Proposition 3.1.3. *Let z be a classical point of $\tilde{\mathcal{E}}^{\mathrm{irr}}$, locally algebraic of weight (k, w) and character ϵ, and with U_v-eigenvalue α. Set $\widetilde{\alpha} = \alpha p^{1+\frac{w-k}{2}}$. Then $V_z|_{G_{F_v}}$ is potentially semistable with Hodge–Tate weights*[4] *$(-\frac{w+k}{2}, -(1+\frac{w-k}{2}))$, and the associated (Frobenius-semisimple) Weil–Deligne representation is of the form*

$$WD(V_z|_{G_{F_v}})^{F-ss} = \left(ur(\widetilde{\beta})^{-1}\epsilon_1 \oplus ur(\widetilde{\alpha})^{-1}\epsilon_2, N\right)$$

for some $\widetilde{\beta} \in \mathbb{C}_p$ (determined by the determinant of V_z) which satisfies $v_p(\widetilde{\alpha}) + v_p(\widetilde{\beta}) = 1 + w$. N is non-zero if and only if $\epsilon_1 = \epsilon_2$ and $\widetilde{\alpha}/\widetilde{\beta} = p^{-1}$. Here we are interpreting characters of $F_v^\times$ as characters of W_{F_v} via the Artin reciprocity map, normalised to take a uniformiser to a geometric Frobenius.

Proof. This follows from the local-global compatibility theorem of Saito [Sai09], as completed by Skinner [Ski09] and T. Liu [Liu12]. Note that α is an eigenvalue for the $\star$-action of U_v on $S(k, w, \epsilon)$ so $\widetilde{\alpha}$ is the corresponding eigenvalue for the standard action of U_v. □

Definition 3.1.4. We say that a classical point z of weight (k, w) is *critical* if $V_z|_{G_{F_v}}$ is decomposable and $v_p(U_v(z)) = k - 1$.

We say that a classical point z is *regular* if it is noncritical and moreover the characters $ur(\widetilde{\beta})^{-1}\epsilon_1$, $ur(\widetilde{\alpha})^{-1}\epsilon_2$ are distinct.

Lemma 3.1.5. *The regular classical points are Zariski dense and accumulation in $\tilde{\mathcal{E}}^{\mathrm{irr}}$.*

Proof. Suppose z is a classical point which fails to be regular. If the failure is because we have $\epsilon_1 = \epsilon_2$ and $\widetilde{\alpha} = \widetilde{\beta}$ then $v_p(\alpha) = \frac{k-1}{2}$. The other possibility is that $V_z|_{G_{F_v}}$ is decomposable and $v_p(U_v(z)) = k - 1$. Any open neighbourhood of z contains a Zariski dense subset of classical points with slope $v_p(\alpha)$ and weights $k' > 2k$. In particular, $v_p(\alpha)$ cannot equal $\frac{k'-1}{2}$ or $k' - 1$, so these points are regular. This shows that the regular classical points are Zariski dense and accumulation. □

In what remains of this section we give details about the triangulations of the p-adic Galois representations for regular classical points of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ and apply the results of [KPX14] to establish the existence of global triangulations for the family of Galois representations over this eigenvariety. Everything here should be unsurprising to experts, but we have written out some details because in the literature it is common to restrict to the semistable case, whilst it is crucial for us to consider classical points which are only potentially semistable.

3.1.1. *Triangulation at v for classical points.* Suppose $z \in \tilde{\mathcal{E}}^{\mathrm{irr}}(L)$ is a regular classical point (with $L/\mathbb{Q}_p$ finite), locally algebraic of weight (k, w) and character ϵ, and with U_v-eigenvalue α. Suppose moreover that $V_z|_{G_{F_v}}$ is indecomposable and potentially crystalline. It follows from Proposition 3.1.3 that this representation becomes crystalline over an Abelian extension of F_v. So the L-Weil–Deligne representation $WD(V_z|_{G_{F_v}})$ has an admissible filtration after extending scalars to the $\mathbb{Q}_p$-algebra $L_\infty =$

[4] We use covariant Dieudonné modules so the cyclotomic character has Hodge–Tate weight -1

$\cup_N L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^N})$ (see [Col08, §4.4]). Moreover, we can explicitly describe this admissible filtered Weil–Deligne module D_z as in [Col08, §4.5]. Indeed, letting e_1 and e_2 be basis vectors with W_{F_v} action given by $ur(\widetilde{\beta})^{-1}\epsilon_1$ and $ur(\widetilde{\alpha})^{-1}\epsilon_2$ respectively, the filtration is given by

$$\mathrm{Fil}^i(L_\infty \otimes_L D_z) = \begin{cases} 0 & \text{if } i > -(1+\frac{w-k}{2}) \\ L_\infty \cdot (\gamma e_1 + e_2) & \text{if } -\frac{w+k}{2} < i \leq -(1+\frac{w-k}{2}) \\ L_\infty \otimes_L D_z & \text{if } i \leq -\frac{w+k}{2} \end{cases}$$

where $\gamma \in L_\infty$ is the (invertible) value of a certain Gauss sum.

It follows from [Col08, Prop. 4.13] that $V_z|_{G_{F_v}}$ is trianguline. To describe the triangulations we adopt the notation of [Col08]. So $\mathscr{R}$ denotes the Robba ring over L and if $\delta : \mathbb{Q}_p^\times \to L^\times$ is a continuous character $\mathscr{R}(\delta)$ denotes that (ϕ, Γ)-module obtained by multiplying the action of ϕ on $\mathscr{R}$ by $\delta(p)$ and the action of $\gamma \in \Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ by $\delta(\chi_{\mathrm{cyc}}(\gamma))$. If V is an L-representation of $G_{\mathbb{Q}_p}$ we denote by $\mathbf{D}_{\mathrm{rig}}(V)$ the slope zero (ϕ, Γ)-module over $\mathscr{R}$ associated to V by [Col08, Prop. 1.7].

Now we have recalled the necessary notation we can recall the precise statement of [Col08, Prop. 4.13]: we have two triangulations

$$0 \to \mathscr{R}(x^{\frac{w+k}{2}} ur(\widetilde{\beta})^{-1}\epsilon_1) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(x^{(1+\frac{w-k}{2})} ur(\widetilde{\alpha})^{-1}\epsilon_2) \to 0$$
$$0 \to \mathscr{R}(x^{\frac{w+k}{2}} ur(\widetilde{\alpha})^{-1}\epsilon_2) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(x^{(1+\frac{w-k}{2})} ur(\widetilde{\beta})^{-1}\epsilon_1) \to 0$$

Now we note that we can rewrite the first of these triangulations as

$$0 \to \mathscr{R}(\delta^{-1} \det \rho(z)) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(\delta(z)) \to 0$$

where we define the continuous character $\delta : F_v^\times \to \Gamma(\tilde{\mathcal{E}}^{\mathrm{irr}}, \mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}})^\times$ by $\delta(p) = U_v^{-1}$ and $\delta|_{\mathcal{O}_{F_v}^\times} = \kappa|_{\{1\}\times\mathcal{O}_{F_v}^\times}$.

Next, we consider the case where $V_z|_{G_{F_v}}$ is not potentially crystalline. Following [Col08, §4.6] we can describe the associated admissible filtered Weil–Deligne module D_z: letting e_2 and e_1 be basis vectors with W_{F_v} action given by $ur(p\widetilde{\alpha})^{-1}\epsilon_2$ and $ur(\widetilde{\alpha})^{-1}\epsilon_2$ respectively, we have $Ne_1 = e_2$, $Ne_2 = 0$, and the filtration is given by

$$\mathrm{Fil}^i(L_\infty \otimes_L D_z) = \begin{cases} 0 & \text{if } i > -(1+\frac{w-k}{2}) \\ L_\infty \cdot (e_1 - \mathscr{L} e_2) & \text{if } -\frac{w+k}{2} < i \leq -(1+\frac{w-k}{2}) \\ L_\infty \otimes_L D_z & \text{if } i \leq -\frac{w+k}{2} \end{cases}$$

for some $\mathscr{L} \in L$.

It follows from [Col08, Prop. 4.18] that $V_z|_{G_{F_v}}$ is trianguline. Indeed, we have a triangulation

$$0 \to \mathscr{R}(x^{\frac{w+k}{2}} |x| ur(\widetilde{\alpha})^{-1}\epsilon_2) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(x^{(1+\frac{w-k}{2})} ur(\widetilde{\alpha})^{-1}\epsilon_2) \to 0$$

which again can be rewritten as

$$0 \to \mathscr{R}(\delta^{-1} \det \rho(z)) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(\delta(z)) \to 0.$$

The final case we have to consider is when $V_z|_{G_{F_v}}$ is decomposable. Then we have $v_p(\alpha) = 0$ (by the regularity assumption on z) and

$$\mathbf{D}_{\mathrm{rig}}(V_z) = \mathscr{R}(x^{\frac{w+k}{2}} ur(\widetilde{\beta})^{-1}\epsilon_1) \oplus \mathscr{R}(x^{(1+\frac{w-k}{2})} ur(\widetilde{\alpha})^{-1}\epsilon_2)$$

We can summarise our discussion in the following:

Proposition 3.1.6. *Suppose $z \in \tilde{\mathcal{E}}^{\mathrm{irr}}$ is a regular classical point. Then there is a triangulation*

$$0 \to \mathscr{R}(\delta^{-1} \det \rho(z)) \to \mathbf{D}_{rig}(V_z) \to \mathscr{R}(\delta(z)) \to 0$$

where we define the continuous character $\delta : F_v^\times \to \Gamma(\tilde{\mathcal{E}}^{\mathrm{irr}}, \mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}})^\times$ by $\delta(p) = U_v^{-1}$ and $\delta|_{\mathcal{O}_{F_v}^\times} = \kappa|_{\{1\}\times\mathcal{O}_{F_v}^\times}$.

Moreover, this triangulation is strict *in the sense of* [KPX14, Def. 6.3.1]

Proof. The only thing remaining to be verified is that the triangulation is strict, which comes down to checking that $\left(\mathbf{D}_{\mathrm{rig}}(V_z)\otimes_{\mathscr{R}}\mathscr{R}(\delta^{-1}\det\rho)^{-1}(z)\right)^{\phi=1,\Gamma=1}$ is one-dimensional over the coefficient field L. Suppose for a contradiction that this space has dimension 2. Then, by [Col08, Prop. 2.1] we have $(\delta^{-1}\det\rho)(z)=x^i\delta(z)$ for some $i\in\mathbb{Z}_{\geq 0}$. Comparing these characters on the intersection of the kernels of ϵ_1 and ϵ_2 we deduce that $i=k-1$. Then these characters are equal if and only if the characters $ur(\widetilde{\beta})^{-1}\epsilon_1$, $ur(\widetilde{\alpha})^{-1}\epsilon_2$ are equal, which contradicts the assumption that z is regular. □

We can now apply the results of [KPX14] to establish the existence of a global triangulation for the family of Galois representations V. In the below statement we adopt the notation of [KPX14, 6.2.1, 6.2.2] so for a rigid space $X/\mathbb{Q}_p$ and a character $\delta_X:\mathbb{Q}_p^\times\to\Gamma(X,\mathcal{O}_X)^\times$ we have a free rank one (ϕ,Γ)-module $\mathscr{R}_X(\delta_X)$ over the sheaf of Robba rings $\mathscr{R}_X$. We also have a rank two (ϕ,Γ)-module $\mathbf{D}_{\mathrm{rig}}(V|_{G_{F_v}})$ over $\mathscr{R}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}$.

Corollary 3.1.7.

(1) $\mathbf{D}_{\mathrm{rig}}(V|_{G_{F_v}})$ *is densely pointwise strictly trianguline in the sense of* [KPX14, Def. 6.3.2]*, with respect to the ordered parameters* $\delta^{-1}\det\rho,\delta$ *and the Zariski dense subset of regular classical points.*

(2) *There are line bundles* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *over* $\tilde{\mathcal{E}}^{\mathrm{irr}}$ *and an exact sequence*

$$0\to\mathscr{R}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}(\delta^{-1}\det\rho)\otimes_{\mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}}\mathscr{L}_1\to\mathbf{D}_{\mathrm{rig}}(V|_{G_{F_v}})\to\mathscr{R}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}(\delta)\otimes_{\mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}}\mathscr{L}_2$$

with the cokernel of the final map vanishing over a Zariski open subset which contains every noncritical classical pont. In particular, this sequence induces a triangulation of $\mathbf{D}_{\mathrm{rig}}(V_z|_{G_{F_v}})$ *at every noncritical classical point.*

Proof. The first assertion follows immediately from Proposition 3.1.6. The existence of the global triangulation and the fact that the cokernel of the final map vanishes over a Zariski open subset containing every regular classical point follows from [KPX14, Cor. 6.3.10]. To apply this Corollary we have to check that there exists an admissible affinoid cover of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ such that the regular classical points are Zariski dense in each member of the cover — this property is the definition of 'Zariski dense' in [KPX14, Def. 6.3.2]. We prefer to reserve the terminology Zariski dense for the standard property that a set of the points is not contained in a proper analytic subset. The (a priori) stronger density statement needed to apply [KPX14, Cor. 6.3.10] follows from [CHJ, Lem. 5.9] (it can be shown that $\tilde{\mathcal{E}}^{\mathrm{irr}}$ has an increasing cover by affinoids as in [CHJ, Rem. 5.10]).

The fact that the global triangulation also induces triangulations at the nonregular (but noncritical) classical points follows from the argument in [KPX14, Prop. 6.4.5]. In fact, one can also show that the triangulation at these points is strict, see the proof of Corollary 3.1.13. □

3.1.2. *p-adic Hodge theoretic properties at* $v'\neq v$. Now we let $v'|p$ be a place of F with $v'\neq v$.

Proposition 3.1.8.

(1) *Let* $z\in\tilde{\mathcal{E}}^{\mathrm{irr}}$*. Then* $V_z|_{G_{F_{v'}}}$ *is potentially semistable with Hodge–Tate weights* $-\frac{w+k_{v'}}{2}$ *and* $-(1+\frac{w-k_{v'}}{2})$.

(2) *The restriction to* $I_{F_{v'}}$ *of the Weil–Deligne representation* $WD(V_z|_{G_{F_{v'}}})$ *depends only on the connected components of* z *in* $\tilde{\mathcal{E}}^{\mathrm{irr}}$.

(3) *If* z *is a classical point with* $U_{v'}(z)=\alpha\neq 0$*, and we set* $\widetilde{\alpha}=\alpha p^{1+\frac{w-k_{v'}}{2}}$ *then the associated (Frobenius-semisimple) Weil–Deligne representation is of the form*

$$WD(V_z|_{G_{F_{v'}}})^{F-ss}=\left(ur(\widetilde{\beta})^{-1}\psi_1\oplus ur(\widetilde{\alpha})^{-1}\psi_2,N\right)$$

for some characters ψ_i *of* $\mathcal{O}_{F,v'}^\times$ *and* $\widetilde{\beta}\in\mathbb{C}_p$ *(determined by the determinant of* V_z*) satisfying* $v_p(\widetilde{\alpha})+v_p(\widetilde{\beta})=1+w$*.* N *is non-zero if and only if* $\psi_1=\psi_2$ *and* $\widetilde{\alpha}/\widetilde{\beta}=p^{-1}$.

Proof. For classical points, the first and third assertions follow from local–global compatibility, as in Proposition 3.1.3. Note that for the third part, a calculation similar to Lemma 2.3.1 shows that we can assume that n is minimal such that $\pi_{v'}^{I'_{v',1,n-1}}$ is non-zero. The rest of the Proposition then follows from

[BC08, Thm. B, Thm. C], which were generalised to reduced quasicompact (and quasiseparated) rigid spaces in [Che, Cor. 3.19], and [BC09, Lem. 7.5.12] (which allows us to identify the Hodge–Tate weights precisely). Note that every point of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ is contained in a quasicompact open subspace with a Zariski dense set of classical points (as in the proof of Corollary 3.1.7, see [CHJ, Lem. 5.9, Rem. 5.10]). □

Definition 3.1.9. We denote by $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$ the Zariski open subspace of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ where $U_{v'}$ is non-zero.

Lemma 3.1.10.

(1) *If $z \in \tilde{\mathcal{E}}^{v'-\mathrm{fs}}$, with $U_{v'}(z) = \alpha \neq 0$, and we set $\widetilde{\alpha} = \alpha p^{1+\frac{w-k_{v'}}{2}}$ then the associated (Frobenius-semisimple) Weil–Deligne representation is of the form*

$$WD(V_z|_{G_{F_{v'}}})^{F-ss} = \Big(ur(\widetilde{\beta})^{-1}\psi_1 \oplus ur(\widetilde{\alpha})^{-1}\psi_2, N\Big)$$

for some characters ψ_i of $\mathcal{O}_{F,v'}^{\times}$ and $\widetilde{\beta} \in \mathbb{C}_p$ (determined by the determinant of V_z) satisfying $v_p(\widetilde{\alpha}) + v_p(\widetilde{\beta}) = 1 + w$.

(2) *$\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$ is a union of connected components of $\tilde{\mathcal{E}}^{\mathrm{irr}}$.*

Proof. We begin by establishing the first part. Suppose $z \in \tilde{\mathcal{E}}^{v'-\mathrm{fs}}$. It follows from the second and third parts of Proposition 3.1.8 that the Weil–Deligne representation at z has a reducible Weil group representation. Since the Weil–Deligne representation varies analytically over $\tilde{\mathcal{E}}^{\mathrm{irr}}$ [BC08, Thm. C], we can consider the characteristic polynomial of a lift of $\mathrm{Frob}_{v'}$ over $\tilde{\mathcal{E}}^{\mathrm{irr}}$. The third part of Proposition 3.1.8 and Zariski density of classical points implies that $p^{1+\frac{w-k_{v'}}{2}}U_{v'}$ is a root of this polynomial over $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$, and this proves that the Weil–Deligne representation has the desired form over all of $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$.

For the second part, we suppose that $z \in \tilde{\mathcal{E}}^{\mathrm{irr}}$ is an element of the Zariski closure of $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$. We need to show that $U_{v'}(z) \neq 0$. The same argument we used to establish the first part shows that the Weil–Deligne representation at z has a reducible Weil group representation with a $\mathrm{Frob}_{v'}$ eigenvalue given by $p^{1+\frac{w-k_{v'}}{2}}U_{v'}(z)$. It follows immediately that $U_{v'}(z) \neq 0$. □

Lemma 3.1.11. *The locus of points $z \in \tilde{\mathcal{E}}^{v'-\mathrm{fs}}$ with $V_z|_{G_{F_{v'}}}$ reducible is equal to the locus of points with $v_p(U_{v'}(z)) = 0$ or $k_{v'} - 1$.*

Moreover, this locus is a union of connected components of $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$.

Proof. It follows from Lemma 3.1.10 and the explicit description of admissible filtered Weil–Deligne modules [Col08, §4.5] that $V_z|_{G_{F_{v'}}}$ is reducible if and only if $v_p(U_{v'}(z)) = 0$ or $k_{v'} - 1$. Since reducibility of a representation is a Zariski closed condition, and the conditions on the slope are open, we deduce that the reducible locus is a union of connected components. □

Definition 3.1.12. We denote the locus of $z \in \tilde{\mathcal{E}}^{v'-\mathrm{fs}}$ with $V_z|_{G_{F_{v'}}}$ irreducible or reducible with $v_p(U_{v'}(z)) = 0$ by $\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}$ (the preceding lemma implies that this is a union of connected components of $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$).

Corollary 3.1.13. *There are line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ over $\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}$, a continuous character $\delta : F_{v'}^{\times} \to \Gamma(\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}, \mathcal{O}_{\tilde{\mathcal{E}}})^{\times}$*

with $\delta(p) = U_{v'}^{-1}$ and a short exact sequence

$$0 \to \mathscr{R}_{\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}}(\delta^{-1}\det\rho) \otimes \mathscr{L}_1 \to \mathbf{D}_{\mathrm{rig}}(V_{\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}}|_{G_{F_{v'}}}) \to \mathscr{R}_{\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}}(\delta) \otimes \mathscr{L}_2 \to 0.$$

Proof. This is proved exactly as Corollary 3.1.7. In this case we claim that we have strict triangulations at every point of $\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}$. The only additional thing we need to check to show that claim is that we have strict triangulations even at a point z where $WD(V_z|_{G_{F_{v'}}})^{F-ss}$ is scalar (in this case, it follows from weak admissibility that $WD(V_z|_{G_{F_{v'}}})$ is not F-semisimple). Setting

$$D = \mathbf{D}_{\mathrm{rig}}(V_z) \otimes_{\mathscr{R}} \mathscr{R}(\delta^{-1}\det\rho)^{-1}(z)$$

we need to check that $D^{\phi=1,\Gamma=1}$ is one-dimensional. It follows from the correspondence between filtered Weil–Deligne modules and potentially semistable (ϕ,Γ)-modules established by [Ber08, Thm. A] that $D[\frac{1}{t}]^{\Gamma=1}$ is two-dimensional with non-trivial unipotent ϕ action, so $D[\frac{1}{t}]^{\phi=1,\Gamma=1}$ is one-dimensional which gives the desired statement. □

We can now give the proof of Theorem 1.2.3.

Proof of Theorem 1.2.3. If $[F:\mathbb{Q}]$ is even, we let D/F be the totally definite quaternion algebra which is split at all finite places. If $[F:\mathbb{Q}]$ is odd, we let D/F be the totally definite quaternion algebra which is split at all finite places except for v_0.

By the Jacquet–Langlands correspondence, we can find a compact open subgroup $K^p \subset (D\otimes_F \mathbb{A}_{F,f}^p)^\times$, an integer $n \geq 1$, a finite order character $\epsilon = \prod_{v|p}\epsilon_v : \prod_{v|p} T_{0,v} \to E_\lambda^\times$ and a Hecke eigenform $f \in H^0(K^p \prod_{v|p} I_{v,1,n-1}, \bigotimes_{v|p}\mathcal{L}(k_v,0,\epsilon_v))$ with associated Galois representation V_f isomorphic to $V_{g,\lambda}$. Moreover, if g is v-(nearly) ordinary at a place $v|p$ we choose f so that its U_v-eigenvalue α_v satisfies $v_p(\alpha_v)=0$. In particular, we have $v_p(\alpha_v) < k_v - 1$ for all $v|p$.

Triviality of the central character of g implies that each character ϵ_v has trivial restriction to the diagonally embedded copy of $\mathcal{O}_v^\times$. The determinant $\det(V_f)$ is the cyclotomic character.

Now we choose an ordering $v_1, v_2, \ldots, v_d$ of the places $v|p$. We can now apply the results of sections 2.7 and 3.1 fixing the place $v = v_1$, together with [PX, Thm. A]. By Lemma 2.7.4, we deduce that we can find an integer $n_1 \geq n$, a finite extension $E_1/\mathbb{Q}_p$, a finite order character $\epsilon_1 = \prod_{v|p}\epsilon_{1,v} : \prod_{v|p} T_{0,v} \to E_1^\times$ and a Hecke eigenform $f_1 \in H^0(K^p \prod_{v|p} I_{v,1,n_1-1}, \mathcal{L}(2,0,\epsilon_{1,v_1}) \bigotimes_{i=2}^d \mathcal{L}(k_{v_i},0,\epsilon_{1,v_i}))$ such that f and f_1 give rise to classical points of a common irreducible component C of $\mathcal{E}(k^{v_1},0)$. Moreover, the U_v-eigenvalues α_v of f_1 satisfy $v(\alpha_v) < k_v - 1$ for all $v|p$. For v_1 this is part of the statement of Lemma 2.7.4. For the other places, this follows from Lemma 3.1.11.

Now we can apply Corollaries 3.1.13 and 3.1.7, together with [PX, Thm. A] to deduce that the validity of the parity conjecture for $V_{g,\lambda}$ is equivalent to the validity of the parity conjecture for V_{f_1}.

We apply this argument $d-1$ more times, and deduce that the validity of the parity conjecture for $V_{g,\lambda}$ is equivalent to the validity of the parity conjecture for V_{f_d}, for a Hecke eigenform

$$f_d \in H^0(K^p \prod_{v|p} I_{v,1,n_d-1}, \bigotimes_{i=1}^d \mathcal{L}(2,0,\epsilon_{d,v_i})).$$

We moreover know that f_d does not generate a one-dimensional representation of $(D\otimes_F \mathbb{A}_F)^\times$ (since it has non-critical slope) and, in the case with $[F:\mathbb{Q}]$ even, f_d is not principal series at v_0. Indeed, each of the one-dimensional families we consider carries a family of Galois representations, and the inertial type of the associated family of Weil–Deligne representations at v_0 is constant. So if f is supercuspidal at v_0, f_d will again be supercuspidal at v_0. If f is a twist of the Steinberg representation at v_0 then the monodromy operator on the family of Weil–Deligne representations is non-zero over a Zariski open subset of the family connecting f to f_1, so the underlying Weil group representation is a sum of two characters with ratio the cyclotomic character. It follows that f_1 is also a twist of Steinberg at v_0, otherwise the local factor at v_0 would be a one-dimensional representation which is not possible for the local factor of a cuspidal automorphic representation. Repeating the argument, we see that in this case f_d is a twist of Steinberg at v_0.

Finally, by the Jacquet–Langlands correspondence and [Nek, Thm. C], the parity conjecture holds for V_{f_d} (note that our Galois representations all have cyclotomic determinant, so the automorphic representation associated to f_d has trivial central character), so the parity conjecture holds for V_f. □

4. The full eigenvariety

4.1. **Notation and preliminaries.** In this section we will consider an eigenvariety for automorphic forms on the quaternion algebra D where the weights are allowed to vary at all places $v|p$. The set-up is very similar to §2.5 so we will be brief.

We set $T_0 = \prod_{v|p} T_{0,v}$, $I_1 = \prod_{v|p} I_{v,1}$ and $I_{1,n} = \prod_{v|p} I_{v,1,n}$, and fix a compact open subgroup $K^p = \prod_{v\nmid p} K_v \subset (\mathcal{O}_D \otimes_F \mathbb{A}_{F,f}^p)^\times$ such that $K = K^p I_1$ is neat.

We let X_p denote the kernel of $N_{F/\mathbb{Q}} : \mathcal{O}_{F,p}^\times \to \mathbb{Z}_p^\times$.

The weight space $\mathcal{W}^{\text{full}}$ is defined by letting

$$\mathcal{W}^{\text{full}}(A) = \{\kappa \in \mathrm{Hom}_{cts}(T_0, A^\times) : \kappa|_{Z(K)} = 1 \text{ and } \kappa|_{X_p} \text{ has finite order.}\}$$

for $\mathbb{Q}_p$-affinoid algebras A. We have $\dim(\mathcal{W}^{\mathrm{full}}) = 1 + d$ and the locally algebraic points (defined as in the partial case) are Zariski dense in $\mathcal{W}^{\mathrm{full}}$.

If the p-adic Leopoldt conjecture holds for F then $Z(K)$ has closure in X_p a finite index subgroup, and the second condition in the definition of the points of weight space is automatic.

For weights $\kappa \in \mathcal{W}^{\mathrm{full}}(A)$ (together with a norm on A which is adapted to κ) and $r \geq r_\kappa$ we get a Banach A-module of distributions $\mathcal{D}_\kappa^r$ equipped with a left action of the monoid $\prod_{v|p} \Delta_v$. We therefore obtain an A-module $H^0(K, \mathcal{D}_\kappa^r)$ equipped with an action of the Hecke algebra $\mathbb{T}$.

We let $U_p = \prod_{v|p} U_v$, which is a compact operator on $H^0(K, \mathcal{D}_\kappa^r)$. We have its spectral variety $\mathcal{Z}^{\mathrm{full}} \to \mathcal{W}^{\mathrm{full}}$ and a coherent sheaf $\mathcal{H}^{\mathrm{full}}$ on $\mathcal{Z}^{\mathrm{full}}$ coming from the modules $H^0(K, \mathcal{D}_\kappa^r)$, equipped with a map $\psi : \mathbb{T} \to \mathrm{End}(\mathcal{H}^{\mathrm{full}})$. We write $\mathcal{E}^{\mathrm{full}}$ for the eigenvariety associated to the eigenvariety datum $(\mathcal{W}^{\mathrm{full}}, \mathcal{Z}^{\mathrm{full}}, \mathcal{H}^{\mathrm{full}}, \mathbb{T}, \psi)$. $\mathcal{E}^{\mathrm{full}}$ is reduced and equidimensional of dimension $1 + d$, with a Zariski dense subset of classical points which we now describe.

Given integers $k = (k_v)_{v|p}$ and w all of the same parity, and a finite order character $\epsilon : T_0 \to E^\times$ which is trivial on $\prod_{v|p}(1 + \varpi_v^n \mathcal{O}_{F_v})^2$, we have an $I_{1,n-1}$-representation $\mathcal{L}(k, w, \epsilon) = \otimes_{v|p} \mathcal{L}_v(k_v, w, \epsilon)$. As in 2.2, k, w and ϵ correspond to a character $\kappa : T_0 \to E^\times$, which we suppose is a point of $\mathcal{W}^{\mathrm{full}}$ (in other words we suppose $\kappa|_{Z(K)} = 1$). We say that κ is locally algebraic of weight (k, w) and character ϵ.

We then obtain a map (as in (2.3.1))

$$\pi : H^0(K, \mathcal{D}_\kappa^r) \to H^0(K^p I_{1,n-1}, \mathcal{L}(k, w, \epsilon)),$$

where for each $v|p$ the action of U_v on the target is the $\star$-*action* defined by multiplying the standard action of U_v by $p^{-(w-k_v+2)/2}$.

Proposition 4.1.1. *For each $v|p$, let $h_v \in \mathbb{Q}_{\geq 0}$ with $h_v < k_v - 1$. The map π induces an isomorphism between the subspaces where U_v acts with slope $\leq h_v$ for all $v|p$.*

Proof. As for Proposition 2.3.2, this can be proved using the method of [Han, Thm. 3.2.5]. □

We say that a point of $\mathcal{E}^{\mathrm{full}}$ is classical if its weight is locally algebraic and the point corresponds to a Hecke eigenvector with non-zero image under the map π above. We say that a classical point (with weight (k, w)) has non-critical slope if the corresponding U_v-eigenvalues have slope $< k_v - 1$ for each $v|p$.

Lemma 4.1.2. *Let $z \in \mathcal{E}^{\mathrm{full}}$ be a classical point of weight (k, w) and character ϵ with non-critical slope. Let $\epsilon = (\epsilon_1, \epsilon_2)$, where ϵ_i is a character of $\mathcal{O}_{F,p}^\times$. If $\epsilon_1|_{\mathcal{O}_{F_v}^\times} = \epsilon_2|_{\mathcal{O}_{F_v}^\times}$ suppose moreover that $v_p(U_v(z)) \neq \frac{k_v - 1}{2}$.*

Then $\mathcal{E}^{\mathrm{full}}$ is étale over $\mathcal{W}^{\mathrm{full}}$ at z.

Proof. This can be proved as in [Che11, Thms. 4.8, 4.10]. Under our assumptions, the Hecke algebra $\mathbb{T}$ acts semisimply on the space of classical automorphic forms of fixed weight and character. We denote the residue field $k(\kappa(z))$ by L. Replacing z by a point (which we also call z) lying over it in $\mathcal{E}_L^{\mathrm{full}}$, it suffices to show that $\mathcal{E}_L^{\mathrm{full}}$ is étale over $\mathcal{W}_L^{\mathrm{full}}$ at z. By the construction of the eigenvariety, we can suppose that we have a geometrically connected (smooth) L-affinoid neighbourhood B of the weight $\kappa(z) \in \mathcal{W}_L^{\mathrm{full}}$ and a finite locally free $\mathcal{O}(B)$-module M, equipped with an $\mathcal{O}(B)$-linear $\mathbb{T}$-action such that:

(1) The affinoid spectrum V of the image of $\mathcal{O}(B) \otimes \mathbb{T}$ in $\mathrm{End}_{\mathcal{O}(B)}(M)$ is an open neighbourhood of $z \in \mathcal{E}^{\mathrm{full}}$.

(2) For each point $x \in B$ and algebraic closure $\overline{k(x)}$ of the residue field $k(x)$ there is an isomorphism of $\overline{k(x)} \otimes \mathbb{T}$-modules

$$M \otimes_{\mathcal{O}(B)} \overline{k(x)} \cong \bigoplus_{y \in \kappa^{-1}(x) \cap V} \bigoplus_{\iota \in \mathrm{Hom}_{k(x)}(k(y), \overline{k(x)})} H^0(K, \mathcal{D}_x^r) \otimes_{k(x)} \overline{k(x)}[\iota \circ \psi_y]$$

where ψ_y denotes the ($k(y)$-valued) system of Hecke eigenvalues associated to y, and $[\iota \circ \psi_y]$ denotes the generalised eigenspace.

(3) $\kappa^{-1}(\kappa(z)) = \{z\}$ and the natural L-algebra surjection $\mathcal{O}(V) \to k(z)$ has a section.

Now we apply the classicality criterion (Proposition 4.1.1), strong multiplicity one, and [Che11, Lem. 4.7] as in [Che11, Thm. 4.10] to deduce that there is a Zariski-dense subset $Z_0 \subset B$ of locally algebraic weights

with residue field L such that κ is étale at each point of $\kappa^{-1}(Z_0)$, and for each $\chi \in Z_0$, $\kappa^{-1}(\chi) \cap V$ consists of a single point with residue field $k(z)$.

We can now show that the map $k(z) \otimes_L \mathcal{O}(B) \to \mathcal{O}(V)$ obtained from assumption (3) is an isomorphism. Since $k(z) \otimes_L \mathcal{O}(B)$ is a normal integral domain, it suffices to show that the map $\pi : \mathrm{Spec}(\mathcal{O}(V)) \to \mathrm{Spec}(k(z) \otimes_L \mathcal{O}(B))$ is birational. By generic flatness, there is a dense open subscheme U of the target such that $\pi|_U$ is finite flat. Since π maps irreducible components surjectively onto irreducible components, $\pi^{-1}(U)$ is a dense open subscheme of the source. Since $Z_0 \cap U$ is non-empty, $\pi|_U$ has degree one and is therefore an isomorphism, which shows that π is birational as desired. In fact, it is not necessary to use the normality of B (as explained to us by Chenevier) — $\mathcal{O}(V)$ is a subalgebra of $\mathrm{End}_{k(z)\otimes_L \mathcal{O}(B)}(k(z) \otimes_L M)$ and each element of $\mathcal{O}(V)$ acts as a scalar in $k(z)$ when we specialise at a weight $\chi \in Z_0$. Since B is reduced and Z_0 is Zariski dense in B, this is enough to conclude that each element of $\mathcal{O}(V)$ acts as a scalar (in $k(z) \otimes_L \mathcal{O}(B)$) on $k(z) \otimes_L M$. □

4.2. **Mapping from partial eigenvarieties to the full eigenvariety.** Now we fix a place $v|p$ and consider the eigenvariety $\mathcal{E}(k^v, w)$ for fixed k^v, w as in section 2.7. For this partial eigenvariety, we fix the level structure $K'_{v'} = I'_{v',1,n-1}$ for each place $v'|p$ with $v' \neq v$, and let $K' = K^p I_{v,1} \prod v' \neq v K'_{v'}$. We denote by $\mathcal{E}(k^v, w)^{U_p - fs} \subset \mathcal{E}(k^v, w)$ the union of irreducible components (with the reduced subspace structure) given by the image of $\cap_{v \neq v'|p} \widetilde{\mathcal{E}}^{v'-fs}$ (see Definition 3.1.9).

We can also construct a spectral variety $\mathcal{Z}^{U_p}(k^v, w)$ and eigenvariety $\mathcal{E}^{U_p}(k^v, w)$ using the compact operator U_p instead of U_v. The closed points of $\mathcal{E}^{U_p}(k^v, w)$ and $\mathcal{E}(k^v, w)^{U_p - fs}$ correspond to the same systems of Hecke eigenvalues, so the following lemma should be no surprise.

Lemma 4.2.1. *There is a canonical isomorphism $\mathcal{E}^{U_p}(k^v, w)^{red} \cong \mathcal{E}(k^v, w)^{U_p - fs}$, over $\mathcal{W}$, compatible with the Hecke operators.*

Proof. Suppose we have a slope datum (U, h) (see [JNa, Defn. 2.3.1]) corresponding to an open subset $\mathcal{Z}^{U_p}_{U,h} \subset \mathcal{Z}^{U_p}(k^v, w)$. We therefore have a slope decomposition

$$H^0(K', \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}^r_U) = M = M_{\leq h} \oplus M_{>h}$$

and a corresponding factorisation $\det(1 - U_v T) = QS$ of the characteristic power series of U_v on M, so we obtain a closed immersion $Z(Q) := \{Q = 0\} \hookrightarrow \mathcal{Z}^{U_v}(k^v, w)|_U$. The module $M_{\leq h}$ defines a coherent sheaf on $Z(Q)$.

Gluing, we obtain a rigid space $\mathcal{Z}^{U_p, U_v}$ equipped with a closed immersion $\mathcal{Z}^{U_p, U_v} \hookrightarrow \mathcal{Z}^{U_v}(k^v, w)$, together with a coherent sheaf $\mathcal{H}^{U_p, U_v}$ on $\mathcal{Z}^{U_p, U_v}$.

Over $\mathcal{Z}^{U_p}_{U,h}$, $\mathcal{E}^{U_p}(k^v, w)^{\mathrm{red}}$ is given by the spectrum of the nilreduction of the affinoid algebra

$$\mathcal{T}_{U,h} = \mathrm{im}(\mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{O}(U) \to \mathrm{End}_{\mathcal{O}(U)} M_{\leq h}).$$

The nilreduction of the eigenvariety associated to the datum $(\mathcal{W}, \mathcal{Z}^{U_p, U_v}, \mathcal{H}^{U_p, U_v}, \mathbb{T}, \psi)$ has the same description, so these two spaces are canonically isomorphic.

On the other hand, we can identify the closed points of $\mathcal{E}(k^v, w)^{U_p - fs}$ and $\mathcal{E}^{U_p}(k^v, w)$, since they correspond to the same systems of Hecke eigenvalues. It follows from [JNb, Thm. 3.2.1], applied to the eigenvariety data $(\mathcal{W}, \mathcal{Z}^{U_p, U_v}, \mathcal{H}^{U_p, U_v}, \mathbb{T}, \psi)$ and $(\mathcal{W}, \mathcal{Z}^{U_v}(k^v, w), \mathcal{H}, \mathbb{T}, \psi)$ that there is a canonical closed immersion $\mathcal{E}^{U_p}(k^v, w)^{red} \hookrightarrow \mathcal{E}(k^v, w)$, which induces the desired isomorphism $\mathcal{E}^{U_p}(k^v, w)^{red} \cong \mathcal{E}(k^v, w)^{U_p - fs}$. □

To compare $\mathcal{E}(k^v, w)^{U_p - fs}$ with $\mathcal{E}^{\mathrm{full}}$, we need to fix a character $\epsilon^v : \prod_{v' \neq v} I_{v',1,n-1}/I'_{v',1,n-1} \to \mathbb{Q}_p^\times$, and consider the eigenvarieties $\mathcal{E}^{U_p}(k^v, w, \epsilon^v)$, $\mathcal{E}(k^v, w, \epsilon^v)$ given by replacing the coherent sheaf $\mathcal{H}$ in the eigenvariety data with the ϵ^v-isotypic piece $\mathcal{H}(\epsilon^v)$. These are unions of connected components in the original eigenvarieties. They are supported over the union of connected components $\mathcal{W}(\epsilon^v) \subset \mathcal{W}$ given by weights κ_v such that $\kappa_v \epsilon^v \chi^v|_{Z(K)} = 1$, where χ^v is the highest weight of $\mathcal{L}^v(k^v, w)$. Note that in general points $\kappa_v \in \mathcal{W}$ satisfy the weaker condition that $\kappa_v \chi^v|_{Z(K')} = 1$.

Definition 4.2.2. We write $\iota_{(k^v, w, \epsilon^v)} : \mathcal{W}(\epsilon^v) \hookrightarrow \mathcal{W}^{\mathrm{full}}$ for the closed immersion defined by

$$\iota_{(k^v, w, \epsilon^v)}(\kappa_v) = \kappa_v \epsilon^v \chi^v$$

where χ^v is the highest weight of $\mathcal{L}^v(k^v, w)$.

Lemma 4.2.3. *There is a canonical closed immersion*

$$\mathcal{E}(k^v, w, \epsilon^v)^{U_p - fs} \hookrightarrow \mathcal{E}^{\mathrm{full}},$$

lying over the map $\iota_{(k^v,w,\epsilon^v)} : \mathcal{W}(\epsilon^v) \hookrightarrow \mathcal{W}^{\mathrm{full}}$*, compatible with the Hecke operators.*

Proof. Applying Lemma 4.2.1, we replace $\mathcal{E}(k^v, w, \epsilon^v)^{U_p - fs}$ in the statement with $\mathcal{E}^{U_p}(k^v, w, \epsilon^v)^{red}$. The classical points of $\mathcal{E}^{U_p}(k^v, w, \epsilon^v)^{red}$ naturally correspond to a subset of the classical points of $\mathcal{E}^{\mathrm{full}}$, and we apply [JNb, Thm. 3.2.1] to obtain the desired closed immersion. □

Corollary 4.2.4. *Let C be an irreducible component of the full eigenvariety $\mathcal{E}^{\mathrm{full}}$. Then C contains a non-critical slope classical point of weight $(2, \dots, 2, 0)$*[5]*.*

Proof. First we note that C contains a non-critical slope classical point x_0 of weight $(k_1, \dots, k_d, w)$ and character $\epsilon = (\epsilon_1, \epsilon_2)$ with w even (since the image of C in weight space is Zariski open). We can moreover assume that the characters $\epsilon_1|_{\mathcal{O}_{F_v}^\times}$ and $\epsilon_2|_{\mathcal{O}_{F_v}^\times}$ are distinct for all $v|p$ (for example by considering weights in a sufficiently small 'boundary poly-annulus' of the weight space). We do this to ensure tht x_0 will be étale over weight space. As in the proof of Theorem 1.2.3, we obtain a sequence $x_1, x_2, \dots, x_d$ of non-critical slope classical points where each x_i has a weight $k(i)$ with $k(i)_1 = k(i)_2 = \cdots k(i)_i = 2$, with the remaining weight components the same as x_0. The v_{i+1} part of the character ϵ changes when we move from x_i to x_{i+1}, but it keeps the property that it has distinct components. So, by Lemma 4.1.2, the map from $\mathcal{E}^{\mathrm{full}}$ to $\mathcal{W}^{\mathrm{full}}$ is étale at each of the points x_i

For each i, x_i and x_{i+1} lie in the image of an irreducible component of $\mathcal{E}(k(i)^{v_{i+1}}, w, \epsilon^{v_{i+1}})^{U_p - fs}$ under the closed immersion of Lemma 4.2.3 (for some choice of $\epsilon^{v_{i+1}}$). It follows that x_i and x_{i+1} lie in a common irreducible component of $\mathcal{E}^{\mathrm{full}}$.

Since the map from $\mathcal{E}^{\mathrm{full}}$ to $\mathcal{W}^{\mathrm{full}}$ is étale at each of the points x_i, x_i is a smooth point of $\mathcal{E}^{\mathrm{full}}$ and is therefore contained in a unique irreducible component of $\mathcal{E}^{\mathrm{full}}$. We deduce that $x_i \in C$ for all i. The point x_d has weight $(2, \dots, 2, w)$. Finally, there is a one-dimensional family of twists by powers of the cyclotomic character connecting x_d to a non-critical slope classical point of weight $(2, \dots, 2, 0)$, which gives the desired point of C. □

Remark 4.2.5. At least for irreducible components C whose associated mod p Galois representation $\overline{\rho}$ is absolutely irreducible, one can deduce cases of the parity conjecture directly from the above corollary, using [PX, Thm. A]. Without this assumption on $\overline{\rho}$, the Galois pseudocharacter over C cannot be automatically lifted to a genuine family of Galois representations over C. This is why our proof of Theorem 1.2.3 instead uses one-dimensional families coming from the partial eigenvarieties, where we can apply Proposition 3.1.1.

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