p-adic Eisenstein series, arithmetic holonomicity criteria, and irrationality of the 2-adic period $\zeta_2(5)$

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June 12, 2020

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$\sum n^{-5}$ and $\zeta(5)$

Archimedean: $\zeta(5) = \sum_{n=1}^{\infty} n^{-5}$, just directly the convergent sum.

The p-adic version of "$\sum n^{-5}$" is to characterize $\zeta_p(5) \in \mathbb{Q}_p$ as the unique constant such that $E_{-4} := \frac{1}{2}\zeta_p(5) + \sum_{N=1}^{\infty} \left(\sum_{n|N,\, p\nmid n} n^{-5} \right) q^N \in \frac{1}{2}\zeta_p(5) + q\mathbb{Q}[[q]]$ is a p-adic modular form. (Serre: A p-adic limit of q-expansions of classical modular forms. Katz: A rigid-analytic section of $\omega^{\otimes(-4)}$ over the locus of ordinary elliptic curves.)

We shall suppose for contradiction that the constant term $\zeta_p(5) \in \mathbb{Q}$, and seek to derive a contradiction by combining the arithmetic and analytic properties of the formal power series $E_{-4}(q) \in \mathbb{Q}[[q]]$ (ultimately coming down to the product formula in number fields).

The Kubota-Leopoldt p-adic zeta function

Theorem. (Calegari, D, Tang, 2020) *The* 2*-adic period* $\zeta_2(5) \notin \mathbb{Q}$.

Here,

$$\zeta_2(5) = \lim_{\text{2-adic}} (1 - 2^{2^k - 5}) \cdot \zeta(5 - 2^k) \in \mathbb{Q}_2.$$

This is a p-adic ($p = 2$) limit of rational numbers. We have $\zeta(1-n) = -B_n/n \in \mathbb{Q}$, where $x/(1-e^{-x}) = \sum_{n=0}^{\infty} B_n x^n/n!$, and Kummer's congruences:

$$(1 - p^{n-1})B_n/n \equiv (1 - p^{m-1})B_m/m \pmod{p^a},$$
$$\text{for } p - 1 \nmid n,\ n \equiv m \mod \phi(p^a);$$

$$(1 - 2^{n-1})B_n/n \equiv (1 - 2^{m-1})B_m/m \pmod{2^a},$$
$$\text{for } 8 \nmid n \text{ and } n \equiv m \mod 2^{a+5}.$$

The Kubota-Leopoldt p-adic zeta function

We can compute this 2-adic number to arbitrary precision:

$$\zeta_2(5) = 2^{-3} + 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{11} + \cdots$$
$$= \overline{1001.11110111101\ldots}_{(2)}.$$

The theorem states that this string of binary digits does not become eventually periodic.
The proof also yields an effective irrationality measure for this constant $\zeta_2(5) \in \mathbb{Q}_2 \setminus \mathbb{Q}$; but I will not discuss this in the talk.

One unusual point in the proof is that it does *not* come along with a "new" rational approximating sequence, and it remains an open problem to construct rapidly convergent rational approximations.

The Kubota-Leopoldt p-adic zeta function

In general, on $s \in \lim_{\leftarrow}(\mathbb{Z}/p^k\mathbb{Z})^\times \cong \mathbb{Z}/(p-1) \times \mathbb{Z}_p$, the Kummer congruences interpolate the rational numbers $(1-p^{n-1})\zeta(1-n)$ [Euler factor at p removed!] to a unique p-adic meromorphic function $\zeta_p(s)$, which is holomorphic apart from a simple pole at $s=1$. It is a p-adic Mellin transform of a Bernoulli measure, in a perfect counterpart to the classical integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^x - 1} x^s \frac{dx}{x}.$$

The special values $\zeta(n)$, resp. $\zeta_p(n)$, at $n \in \mathbb{N}$ are real, resp. p-adic periods of mixed Tate motives over $\mathbb{Z}$, via Chen's, resp. Coleman's iterated integrals.

Transcendence of zeta values

We have $\zeta(2n) \in \pi^{2n} \cdot \mathbb{Q}$ and $\zeta_p(2n) = 0$. It is conjectured that

$$\pi, \zeta(3), \zeta(5), \zeta(7), \ldots \subset \mathbb{R},$$

and likewise for

$$\zeta_p(3), \zeta_p(5), \zeta_p(7), \ldots \subset \mathbb{Q}_p,$$

are transcendental and algebraically independent. (More generally: that the only algebraic relations among periods - real or *p*-adic - are the ones "of motivic origin"). Essentially our present-day state of knowledge reduces to:

$\pi \notin \overline{\mathbb{Q}}$ (Lindemann, 1882); $\zeta(3) \notin \mathbb{Q}$ (Apéry, 1978); and $\zeta_2(3), \zeta_3(3) \notin \mathbb{Q}$ (Calegari, 2005).

Transcendence of zeta values

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Our goal in this talk is to add $\zeta_2(5)$ to this list, as well as a possible continuation of the method. May it be employed in the Archimedean world, such as notably for the Catalan constant $C = L(2, \chi_4)$?

Although there are further (celebrated) results to the effect that infinitely many odd zeta values $\zeta(2k+1)$ are irrational (Rivoal, 2000), and at least one among the four numbers $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ are irrational (Zudilin, 2001), we focus here on pure irrationality statements.

Apéry

In 1978 at a conference in Luminy, Apéry stunned mathematicians by proving the long-standing conjecture that $\zeta(3)$ is irrational. He did this by displaying (without much explanation) an explicit sequence $a_n/b_n \approx \zeta(3)$ of rapidly convergent rational approximations:

$$b_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \in \mathbb{Z},$$

$$a_n := \sum_{k=0}^{n} \left\{ \binom{n}{k}^2 \binom{n+k}{k}^2 \Big(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \Big) \right\} \in \frac{1}{2[1,\ldots,n]^3}\mathbb{Z},$$

with $|a_n - b_n\zeta(3)| < (\sqrt{2}-1)^{4n}$.

Apéry

Thus $|a_n - b_n\zeta(3)| < (\sqrt{2}-1)^{4n}$, but this linear form is in $\frac{1}{2[1,\ldots,n]^3}\cdot\mathbb{Z} + \zeta(3)\cdot\mathbb{Z}$. Since $e^3\cdot(\sqrt{2}-1)^4 < 1$, this certifies an irrationality proof for $\zeta(3)$.

The explanation that Apéry *did* provide (whose *ad hoc* verification is the most difficult part of his proof) is that these sequences a_n and b_n are *holonomic*: their generating functions $U(x) := \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Q}[[x]]$ and $V(x) := \sum_{n=0}^{\infty} b_n x^n \in \mathbb{Z}[[x]]$ are the solutions regular at the origin $x = 0$ to the common linear ODE $L(f) = 0$, where

$$L := \frac{d}{dx}\left\{x^2(x^2-34x+1)\left(\frac{d}{dx}\right)^3 + x(3-153x+6x^2)\left(\frac{d}{dx}\right)^2 + (1-112x+7x^2)\frac{d}{dx} + x - 5\right\}$$

with the respective initial conditions

$$U(0) = 6, U'(0) = 0; \quad V(0) = 1, V'(0) = 5.$$

Apéry limits and the overconvergence characterization

Then $\lambda = \zeta(3)$ is characterized as the unique constant for which the power series (solution to $L(f) = 0$ holomorphic at the origin)

$$U(x) - \lambda V(x)$$

converges further than for any other value: for $\lambda = \zeta(3)$ this power series converges up to the "larger" singularity $(\sqrt{2}+1)^4$ of the linear differential operator L; for $\lambda \neq \zeta(3)$ it only converges up to the "smaller" singularity $(\sqrt{2}-1)^4$ (those are the two roots of $x^2 - 34x + 1$).

We say that the constant $\zeta(3)$ is characterized by an overconvergence.

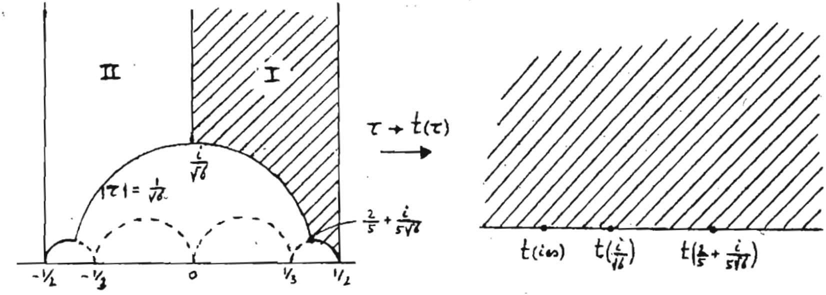
Beukers and modular forms for the level $\Gamma_1(6)$

Beukers (1987, *Irrationality proofs using modular forms*) found an insightful interpretation of Apéry's sequences, via an analysis with modular forms on the curve $X_1(6)^+ := X_1(6)/w_6$. Here $w_6 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}$ is the Fricke involution $\tau \mapsto -1/(6\tau)$ of the upper half plane. This modular curve $X_1(6)^+$ is rational with Hauptmodul

$$x := \left(\frac{\eta(6\tau)\eta(\tau)}{\eta(3\tau)\eta(2\tau)}\right)^{12} = q - 12q^2 + 66q^3 - 220q^4 + \cdots ,$$

mapping the cusps $i\infty \mapsto 0$, $1/2 \mapsto \infty$, the two elliptic points $i/\sqrt{6} \mapsto (\sqrt{2}-1)^4$ and $\frac{2}{5} + i\frac{1}{5\sqrt{6}} \mapsto (\sqrt{2}+1)^4$, and the domain "I+II" conformally onto the slit plane $\mathbb{C} \setminus [(\sqrt{2}-1)^4, \infty)$

Beukers and the Hauptmodul for $X_1(6)^+$



Graphic taken from Beukers's paper; we write x instead of t; the three marked values are the singularities $0, (\sqrt{2}-1)^4$ and $(\sqrt{2}+1)^4$ of Apéry's differential equation.

Beukers and Apéry with $X_1(6)^+$

Apéry's generating function is

$$\sum_{n=0}^{\infty}(a_n - b_n\zeta(3))x^n = E(x)\cdot(f(x)-\zeta(3)),$$

where $q=\exp(2\pi i\tau) = x + 12x^2 + 222x^3 + \cdots \in x + x^2\mathbb{Z}[[x]]$ has been formally inverted, and [these combinations are essentially uniquely determined by the requirements below]

$$24E(\tau) := -5(E_2(\tau)-6E_2(6\tau)) + (2E_2(2\tau)-3E_2(3\tau)) \in M_2(\Gamma_1(6)) \\ \in \mathbb{Z}[[q]] = \mathbb{Z}[[x]], \quad w_6^* E(\tau) = -6\tau^2 E(\tau),$$

and $f(\tau) := \sum_{n=1}^{\infty} \frac{f_n}{n^3} q^n \in \mathbb{Q}[[q]] = \mathbb{Q}[[x]]$ with $F(\tau) = \sum_{n=1}^{\infty} f_n q^n \in \mathbb{Z}[[q]]$ the Fourier expansion of the weight-4 cusp form

$$F(\tau) := \frac{1}{40}\big(E_4(\tau) - 6^2E_4(6\tau)\big) - \frac{7}{40}\big(2^2E_4(2\tau) - 3^2E_4(3\tau)\big) \in S_4(\Gamma_1(6)).$$

Beukers and Apéry with $X_1(6)^+$

We have $w_6^* F(\tau) = -6^2\tau^4 F(\tau)$, which "Hecke's lemma" (Prop. 1.2 in Beukers) converts to

$$w_6^*(f(\tau) - L(F,3)) = -\frac{1}{6\tau^2}(f(\tau) - L(F,3))$$

on the iterated triple integral f of F, and this modularity relation characterizes the constant $L(F,3)$, as an Eichler period. A simple computation shows
$L(F,s) = 6(1 - 6^{2-s} - 7\cdot 2^{2-s} + 7\cdot 3^{2-s})\zeta(s)\zeta(s-3)$, and
$L(F,3) = \zeta(3)$. Multiplying by the complementary relation
$w_6^* E(\tau) = -6\tau^2 E(\tau)$ exactly cancels out the automorphy factors:

> $E\cdot(f - \zeta(3))$ *has trivial monodromy around the elliptic point* $i/\sqrt{6}$ *fixed by the Fricke involution* w_6.

$\zeta(3)$ by overconvergence; conclusion by a trivial arithmetic rationality criterion

By the picture, that means exactly that the power series germ $H(x) := E(x) \cdot (f(x) - \zeta(3))$, which is *a priori* analytic on the fairly small domain $\mathbb{C} \setminus [(\sqrt{2}-1)^4, \infty)$ in the x-plane, is in fact analytic on the fairly large domain $\mathbb{C} \setminus [(\sqrt{2}+1)^4, \infty)$.

If now $\zeta(3) \in \mathbb{Q}$, then $H(x) = \sum_{n=0}^{\infty} c_n x^n$ would have rational coefficients: $c_n = a_n - b_n\zeta(3) \in \frac{1}{2[1,\ldots,n]^3}\mathbb{Z} + \zeta(3)\mathbb{Z}$, and the following properties which are contradictory:

- $\tau(H(x)) := \lim_{p_0 \to \infty} \limsup_{n \in \mathbb{N}} \frac{1}{n} \sum_{p \geq p_0} \max_{i=0}^{n} \log |c_i|_p = 3$;
- $H(x)$ is analytic (convergent) on the complex disk $|x| < (\sqrt{2}+1)^4$, and on the unit p-adic disk $|x|_p < 1$ for every prime p;
- the total fuel (arithmetic degree of the domain of convergence) $\sum_{\text{all } v} \log R_v = \log(\sqrt{2}+1)^4 = 3.525\ldots > 3 = \tau(H)$, yet $H(x) \notin \mathbb{Q}[x]$.

Analyticity on a larger domain than the disk of convergence

- *This proof only used the disk of convergence,* $|x| < (\sqrt{2}+1)^4$, *recovering the Apéry irrationality measure* $\mu_0 = \frac{8\log 1+\sqrt{2}}{4\log(1+\sqrt{2})-3} = 13.417\ldots$ *on* $\zeta(3)$. *May this be improved, and how much, on using that in fact* $f(x)$ *is analytic on the larger domain* $\mathbb{C} \setminus [(\sqrt{2}+1)^4, \infty)$, *whose conformal mapping radius is four times as high as the radius of its largest contained disk?* (Record: $\mu_0 = 5.513\ldots$, by Rhin and Viola)
- *Further, may we improve the numerics by using also analyticity of the pull-back of* H *on a suitable finite covering of* $X_1(6)^+$, *of a higher genus?*

First example: Zudilin's determinantal criterion

With the above linear forms $r_n = a_n - b_n\xi$ (we had $\xi = \zeta(3)$), we have used the following trivial criterion for irrationality of ξ. Suppose $r_n = O(\varepsilon^n)$ and $\delta_n a_n, \delta_n b_n \in \mathbb{Z}$, with the δ_n positive integers with exponential growth rate $\Delta = \lim_n(\delta_n)^{1/n} \in [1, \infty)$.

> *Then if $\varepsilon\Delta < 1$, and $r_n \neq 0$ infinitely often, ξ must be irrational.*

In terms of the generating power series $f(x) := \sum_{n=0}^{\infty} r_n x^n$, the Archimedean convergence radius $\rho_\infty(f) \geq 1/\varepsilon$. Let us assume additionally that $\delta_n \mid \delta_{n+1}$ for each n, as in the application with $\delta_n = 2[1, \ldots, n]^3$. We may then rephrase our input as a trivial arithmetic algebraization theorem:
If $f(x) \in \mathbb{Q}[[x]]$ has convergence radii satisfying $\sum_v \log \rho_v(f) > \tau(f)$, then $f \in \mathbb{Q}[x]$ is a polynomial.

First example: Zudilin's determinantal criterion

Reference: W. Zudilin, *A determinantal approach to irrationality*, Constr. Approx. (2017).
This trivial criterion sufficed for Apéry's $\zeta(3)$. It also sufficed in Calegari's 2-adic and 3-adic analog of Apéry's theorem. But it will not be enough for our $\zeta_2(5)$ irrationality proof. As a warm up, here is a simpler situation which may be also applied to exponentially divergent linear forms $r_n = a_n - b_n\xi$ (case $\varepsilon > 1$; note that $\Delta \geq 1$ in any case).

Suppose additionally that $\delta_n \mid \delta_{n+1}$ for all n, and that the $r_n = \int p(t)^n \, d\mu(t)$, where $p(t) \geq 0$ and $d\mu$ is a (non-negative) measure on $\mathbb{R}$. If $\varepsilon\Delta^{3/2} < 4$, then ξ is irrational.

This is more clearly seen on the level of the generating function: $f(x) = \sum_{n=0}^{\infty} r_n x^n \in \mathbb{Q}[[x]]$ has $\tau(f) \leq \log \Delta$ and has analytic continuation to the larger domain $\mathbb{C} \setminus (-\infty, -1/\varepsilon]$, of conformal mapping radius $4/\varepsilon$.

First example: Zudilin's determinantal criterion

What he really proves is that $f \in \mathbb{Q}[[x]]$ is rational under the conditions:

- $f\left(z/(1-\frac{z}{4r})^2\right) \in \mathbb{C}[[z]]$ is convergent on the complex disk $|z| < 4r$ (that means precisely f is holomorphic on $\mathbb{C} \setminus (-\infty, -r]$;
- $f(x)$ is convergent on $|x|_p < 1$ for every prime p; and
- the domain of analyticity is large with respect to the denominators: $\log(4r) > \frac{3}{2}\log\Delta$.

But the example of $f(x) = \log(1+x)$, with $r = 1$ and $\Delta = e$, demonstrates that the coefficient $3/2$ may not be dropped to below the value $\log 4$. Observe nonetheless that this function is *holonomic*: it fulfills a linear ODE with polynomial coefficients. We will show that *for holonomicity*, the coefficient $3/2$ can be reduced to the best-possible value 1. (*Q: What is the optimal coefficient in Zudilin's criterion for rationality?*)

The arithmetic holonomicity theorem

We can formulate it over any global field K. Normalize the absolute values $|\cdot|_v$, $v \in M_K$ in the usual way ($|\alpha|_v$ is the "module" reflecting the change in Haar measure of K_v^+ under $x \mapsto \alpha x$), so that the product formula holds. We extend our notation to $f(\mathbf{x}) = \sum a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in K[[x_1, \ldots, x_d]]$. For $V \subset M_K$, let

$$h_K([\alpha_0 : \ldots : \alpha_N]) := \sum_{v \in M_K} \max_{0 \le i \le N} \log |\alpha_i|_v$$

$$h_K^{(V)}([\alpha_0 : \ldots : \alpha_N]) := \sum_{v \in M_K \setminus V} \max_{0 \le i \le N} \log |\alpha_i|_v$$

$$|\mathbf{n}| := n_1 + \cdots + n_d; \quad h_K(f) := \limsup_{n \in \mathbb{N}} \frac{1}{n} h_K((a_{\mathbf{n}})_{\mathbf{n} : |\mathbf{n}| \le n})$$

$$h_K^{(V)}(f) := \limsup_{n \in \mathbb{N}} \frac{1}{n} h_K^{(V)}((a_{\mathbf{n}})_{\mathbf{n} : |\mathbf{n}| \le n})$$

$$\tau_K(f) := \inf_{V : \# V < \infty} h_K^{(V)}(f).$$

The arithmetic holonomicity theorem: the invariant $\tau(f)$

Thus $\tau_K(f)$ is the 'truly arithmetic' part of the height $h_K(f)$:

$$h_K(f) = \sum_{\text{all} v \in M_K} \log^+ \frac{1}{\rho_v(f)} + \tau_K(f),$$

where $\rho_v(f)$ is the v-adic convergence radius of f. The condition $\tau_K(f) = 0$ is a mild quantitative strengthening of S-integral coefficients. It is also known as *A-analyticity* in Bost's more general (symmetrical with respect to either axis for the formal graph of f) setting of formal subschemes of arithmetic schemes. In the Grothendieck-Katz algebraicity conjecture, the condition of a.e. vanishing p-curvatures of an arithmetic differential equation $L(f) = 0$ is equivalent to the existence of a basis of "$\tau(f) = 0$" solutions at some (equivalently, every) non-singular point. Little appears known about the invariant $\tau(f)$ of *holonomic* power series f: is it always a rational number? if it is not zero, how small can it be in terms of the number of singular points?

The arithmetic holonomicity theorem

$\mathbb{C}_v$: the completion of an algebraic closure of K_v.

$$D_d(\mathbf{0}, R_v) := \{\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}_v^d \;:\; \max_{1 \le i \le d} |z_i|_v < R_v\}$$

For each $v \in M_K$*, suppose given a radius* $R_v > 0$ *and a holomorphic mapping* $\mathbf{x}_v(\mathbf{z}) : D_d(\mathbf{0}, R_v) \to \mathbb{C}_v^d$*, normalized by* $\mathbf{x}_v(z) = \mathbf{z} + O(\mathbf{z}^2)$*, and such that* $\mathbf{x}_v(\mathbf{z}) = \mathbf{z}$ *for all but finitely many* v*. Let* $f(x_1, \ldots, x_d) \in K[[\mathbf{x}]]$ *be such that, for every* $v \in M_K$*, the formal power series*

$$f(\mathbf{x}_v(\mathbf{z})) \in \mathbb{C}_v[[\mathbf{z}]]$$

is the germ of a v*-adic meromorphic function on the polydisk* $D_d(\mathbf{0}, R_v)$*.* Observe that if $\sum_{v \in M_K} \log R_v \le \tau$, the cardinal number of such $f \in K[[\mathbf{x}]]$ having $\tau_K(f) = \tau$ is the continuum.

The arithmetic holonomicity theorem

Theorem. *Under the sharp arithmetic degree condition*

$$\sum_{v \in M_K} \log R_v > \tau_K(f),$$

the power series $f(\mathbf{x})$ is holonomic: it satisfies a linear homogeneous differential equation $L(f) = 0$, where

$$L = \sum_{\mathbf{i}\colon i_1,\ldots,i_d < r} a_{\mathbf{i}}(\mathbf{x}) \partial_1^{i_1} \cdots \partial_d^{i_d}, \quad \partial_i := x_i \frac{\partial}{\partial x_i}, \quad a_{\mathbf{i}}(\mathbf{x}) \in K(\mathbf{x}).$$

More precisely, with $S_v := \sup_{|\mathbf{z}|_v = R_v} |\mathbf{x}_v(\mathbf{z})|_v$, there is then an $inhomogeneous$ *differential equation $L(f) \in K[x]$, for some linear differential operator L as above having*

$$r \leq \frac{\sum_v \log^+ S_v}{\sum_v \log R_v - \tau_K(f)}$$

The gist of Schinzel-Zassenhaus

Let $K = \mathbb{Q}$, $d = 1$ and $f(x) \in \mathbb{Z}[[x]]$, hence we may take $R_p = 1$ and $x_p(z) = z$ for all primes p. It follows as a special case that if $f(x(z))$ is analytic for an $x(z) = z + \cdots : D(0, R) \to D(0, S)$ with an $R > 1$ and $\frac{\log^+ S}{\log R} < 2$, then $f(x) \in \mathbb{Q}(x) \cap \mathbb{Z}[[x]]$ is a rational function.

Apply with

$$f(x) := \sqrt{\prod_{\alpha :\, P(\alpha)=0} (1 - \alpha^2 X)(1 - \alpha^4 X)} \in \mathbb{Z}[[x]]$$

for $P(x) \in \mathbb{Z}[x]$ a degree-n polynomial with $P(0) = 1$. If $\min_{\alpha :\, P(\alpha)=0} \log |\alpha| \geq 1 - c/n$ with a certain absolute $c > 0$ small enough, a theorem of Dubinin supplies such a $x(z)$ with certain radii $R = 1 + c_1/n$ and $S = 1 + (1.5c_1)/n$. Consequently, $\min_{\alpha :\, P(\alpha)=0} \log |\alpha| < 1 - c/n$ unless $\pm P(x)$ is not a product of cyclotomic polynomials and a power of x. Details and extensions in `ArXiv:1912.12545v1`, with the precise constant $c = (\log 2)/4$.

Proof of the holonomicity theorem

We first show how to reduce to the following linear dependency criterion of André, VIII 1.6 from his book *G-Functions and Geometry*. (NB: The set $V \subset M_K$ in *loc. cit.* must be assumed to have $\#V < \infty$.)

Suppose $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) \in K[[\mathbf{x}]]$ *have* $f(\mathbf{x}_v(\mathbf{z}))$ *meromorphic on* $D_d(\mathbf{0}, R_v)$*. If for some real parameter* $\kappa \in \mathbb{R}^{>0}$ *the inequality*

$$\sum_{v \in M_K} \log R_v > \tau_K(f_1, \ldots, f_m) + \kappa \cdot h_K(f_1, \ldots, f_m) + \Big(\frac{1}{m}(1 + 1/\kappa)\Big)^{1/d} \cdot \sum_{v \in M_K} \log^+ S_v$$

is fulfilled, then $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ *are* $K(\mathbf{x})$*-linearly dependent.*

Proof of the holonomicity theorem

Condition in the general André theorem:

$$\sum_{v \in M_K} \log R_v > \tau_K + \kappa \cdot h_K + \left(\frac{1}{m}\left(1 + 1/\kappa\right)\right)^{1/d} \cdot \sum_{v \in M_K} \log^+ S_v$$

Our basic goal, in our particular situation, is to delete the terms involving the parameter κ. (Optimization in κ is not good enough for our irrationality proof.) Before we take it up in a moment, some intuitive guidelines:

- $\sum_{v \in M_K} \log R_v > \tau_K$ is the essential positivity condition;
- the proof is by an auxiliary (Siegel lemma) linear form $Q_1 f_1 + \cdots + Q_m f_m$ vanishing highly at the origin, with polynomials Q_i of heights bounded by κh_K and degrees bounded by $\left(\frac{1}{m}\left(1 + 1/\kappa\right)\right)^{1/d}$.
- $\log^+ S_v$ term arises from estimating $\sup_{|\mathbf{z}|_v = R_v} |Q_i(\mathbf{x}_v(\mathbf{z}))|$.

Proof of the holonomicity theorem: the bootstrapping

Observe that the qualitative form of the holonomicity theorem is already an easy consequence of André's criterion. Apply with $f_i := \partial_1^i f$; then $\tau_K(f_1, \ldots, f_m) = \tau_K(f)$, $h_K(f_1, \ldots, f_m) = h_K(f)$, and by the chain rule, the $f_i(\mathbf{x}_v(\mathbf{z}))$ are still meromorphic on $D_d(\mathbf{0}, R_v)$. Now first let $m \to \infty$ with respect to κ, and then $\kappa \to 0$. We get that if $\sum_v \log R_v > \tau_K(f)$, the derivatives cannot be $K(\mathbf{x})$-linearly independent.

The crucial point of the application is the precise estimate on the rank r in the $d = 1$ case. The trick is to multiply the blocks of variables and consider, instead of the original variety $\mathbb{A}_K^d$, its high power $(\mathbb{A}_K^d)^s = \mathbb{A}_K^{sd}$, and let $s \to \infty$ in the limit.

This is the only reason that we allow multiple variables. Our ultimate application will be on the rational modular curve $X_0(2)$, but the proof goes through a Diophantine approximation on a high enough power $X_0(2)^s$.

Proof of the holonomicity theorem: the bootstrapping

So suppose with some

$$r > \frac{\sum_v \log^+ S_v}{\sum_v \log R_v - \tau_K(f)}$$

that the r^d derivatives $\partial_{\mathbf{i}} f(\mathbf{x})$, $0 \leq i_1, \ldots, i_d < r$, are $K(\mathbf{x})$-linearly independent. Thus

$$(\star) \qquad \sum_v \log R_v > \tau_K(f) + \frac{1}{r} \sum_v \log^+ S_v.$$

Apply the André's linear dependency criterion to the $m = r^{ds}$ *disjoint variables* [crucial point!] products

$$\partial_{\mathbf{i}_1} f(\mathbf{x}_1) \cdots \partial_{\mathbf{i}_s} f(\mathbf{x}_s) \in K[[\mathbf{x}_1, \ldots, \mathbf{x}_s]].$$

By $(\star)$, *there will be a small enough* $\kappa > 0$ *and a large enough* $s \in \mathbb{N}$ *such that the inequality in André's criterion is met...*

Proof of the holonomicity theorem: the bootstrapping

…in light of the following key observation:

Lemma. *Consider two non-zero formal power series $f(\mathbf{x}) \in K[[\mathbf{x}]] \setminus \{0\}$ and $g(\mathbf{y}) \in K[[\mathbf{y}]] \setminus \{0\}$ in the disjoint blocks of variables $\mathbf{x}$ and $\mathbf{y}$. Then the product series*

$$H(\mathbf{x}, \mathbf{y}) := f(\mathbf{x})g(\mathbf{y}) \in K[[\mathbf{x}, \mathbf{y}]]$$

has

$$h_K(H) = h_K(f, g) \quad \textit{and} \quad \tau_K(H) = \tau_K(f, g).$$

(Multiplication has no 'carries' when the variables are disjoint.)
In sharp contrast: in general (unless $\tau_K(f) = 0$: Bost's *A-analytic condition*), both $h_K(f^2) > h_K(f)$ and $\tau_K(f^2) > \tau_K(f)$ are strict. The bound $\tau(f^n) \leq \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\tau(f)$ is asymptotically sharp for e.g. $f(x) = \log(1 - x)$.

Proof of André's criterion: the Siegel lemma part

Fix parameters $\alpha \in \mathbb{N}$ and $\kappa \in \mathbb{R}^{>0}$. Asymptotically in α for the fixed $(f_1, \dots, f_m)$, there exists an m-tuple of polynomials $Q_1, \dots, Q_m \in K[\mathbf{x}]$, not all zero, such that:

(i) $\max_{i=1}^m \deg Q_i \leq \left(\frac{1}{m}\left(1 + \frac{1}{\kappa}\right)\right)^{\frac{1}{d}} \alpha + o(\alpha)$;

(ii) $\max_{i=1}^m h_K(Q_i) \leq \kappa \cdot h_K(f_1, \dots, f_m) \cdot \alpha + o(\alpha)$;

(iii) $Q_1 f_1 + \cdots + Q_m f_m \in (x_1, \dots, x_d)^\alpha$ *(order of vanishing $\geq \alpha$ at the origin).*

We have $\binom{\alpha+d-1}{d} \sim \alpha^d/d!$ linear equations in the $m\binom{N+d}{d} \sim mN^d/d!$ unknown coefficients. The degree parameter choice

$$N \sim \left(\frac{1}{m}\left(1 + \frac{1}{\kappa}\right)\right)^{\frac{1}{d}} \alpha > m^{-1/d} \alpha$$

insures that the solution space is non-zero, and brings in a Dirichlet exponent of $\sim \kappa\alpha$ in Siegel's lemma.

Proof of André's criterion: the extrapolation

We next push this Siegel lemma approximation to a full $K(\mathbf{x})$-linear dependency

$$U := Q_1 f_1 + \cdots + Q_m f_m = 0.$$

If this power series $U(\mathbf{x})$ is not zero, let $\beta \geq \alpha$ be the minimum degree of a non-vanishing term, and choose a multi-index $\mathbf{k} \in \mathbb{N}_0^d$ with

$$\eta := \frac{1}{\mathbf{k}!} \frac{\partial^{\mathbf{k}} U(\mathbf{x})}{\partial \mathbf{x}^{\mathbf{k}}}\Big|_{\mathbf{x}=\mathbf{0}} = \frac{1}{\mathbf{k}!} \frac{\partial^{\mathbf{k}} U(\mathbf{x}_v(\mathbf{z}))}{\partial \mathbf{z}^{\mathbf{k}}}\Big|_{\mathbf{z}=\mathbf{0}} \neq 0.$$

We choose a large enough finite subset $V \subset M_K$ of the places, and estimate the prime-to-V part of η trivially by the Liouville estimate:

$$\sum_{v \notin V} \log |\eta|_v \leq \max_{1 \leq i \leq m} h_K^{(V)}(Q_i) + \beta \cdot h_K^{(V)}(f_1, \ldots, f_m) + o(\beta). \quad (1)$$

Proof of André's criterion: the extrapolation

The second term in this estimate (1) is proportional-asymptotic to the inevitable "denominators part" $\tau_K(f_1,\ldots,f_m)$.
At the finitely many places $v \in V$, we use a stronger estimate coming by way of the v-adic analytic representation

$$q(\mathbf{z})U(\mathbf{x}_v(\mathbf{z})) = Q_1(\mathbf{x}_v(\mathbf{z}))h_1(\mathbf{z}) + \cdots + Q_m(\mathbf{x}_v(\mathbf{z}))h_m(\mathbf{z}) \in \mathbb{C}_v[[\mathbf{z}]],$$

$$q(\mathbf{0}) = 1; \quad \text{thus still } \eta = \frac{1}{\mathbf{k}!}\frac{\partial^{\mathbf{k}}(q(\mathbf{z})U(\mathbf{x}_v(\mathbf{z})))}{\partial \mathbf{z}^{\mathbf{k}}}\Big|_{\mathbf{z}=\mathbf{0}},$$

where now $\mathbf{x}_v(\mathbf{z}), q(\mathbf{z})$ and $h_i(\mathbf{z})$ are holomorphic on the v-adic polydisk $\|\mathbf{z}\|_v := \max_{i=1}^d |\mathbf{z}|_v \leq R_v$. (Shrink the R_v a little bit if necessary.)

Proof of André's criterion: the extrapolation

On the boundary torus $\|\mathbf{z}\|_v = R_v$ we have the estimate

$$\log|q(\mathbf{z})U(\mathbf{x}_v(\mathbf{z}))|_v \le \log\left(m\binom{N+d}{d}\right) + N\log^+ S_v$$
$$+ \max_{1\le i\le m} \sup_{\|\mathbf{z}\|_v=R_v} \log^+|h_i(\mathbf{z})|_v + \sup_{\|\mathbf{z}\|_v=R_v} \log^+|q(\mathbf{z})|_v + \max_{1\le i\le m} h_v(Q_i).$$

Here

$$N := \max_{1\le i\le m} \deg Q_i \le \left(\frac{1}{m}\left(1+\frac{1}{\kappa}\right)\right)^{\frac{1}{d}}\beta + o(\beta)$$

in Siegel's lemma. We use this to estimate

$$\eta = \frac{1}{(2\pi i)^d}\int_{\|\mathbf{z}\|=R_v} q(\mathbf{z})U(\mathbf{x}_v(\mathbf{z}))\frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{k}}}$$

[this Cauchy integral formula is for the Archimedean case; analogous appeal to the maximum principle in the ultrametric case]

Proof of André's criterion: the extrapolation

At this point we use Schwartz's lemma: as the integrand $q(\mathbf{z})U(\mathbf{x}(\mathbf{z}))/\mathbf{z}^{\mathbf{k}}$ is holomorphic, and $|\mathbf{k}| = \beta$, the Cauchy estimate yields

$$\log|\eta|_v \leq \left(\frac{1}{m}\left(1+\frac{1}{\kappa}\right)\right)^{\frac{1}{d}}\beta + \max_{1\leq i\leq m} h_v(Q_i) - \beta\log R_v + o(\beta). \quad (2)$$

Now if $V \subset M_K$ and $\alpha \leq \beta$ are large enough, this contradicts André's inequality upon adding (1) (on $M_K \setminus V$) to (2) over all $v \in V$. Thus André's condition forces identical vanishing $Q_1 f_1 + \cdots + Q_m f_m \equiv 0$, completing the proof of the André's criterion, and of the holonomicity theorem.

p-adic Eisenstein series

The p-adic zeta function is best understood as the constant term in the q-expansion of a p-adic (rigid analytic) family of Eisenstein series, whose non-constant Fourier coefficients are just divisor-sum functions. This was Serre's approach to the p-adic Kubota-Leopoldt L-function, to use the whole Eisenstein family for bootstrapping analytic properties from the non-constant terms to the constant term.

Concretely, we shall start with the classical (algebraic, with p-Euler factor removed) weight-$2k$ Eisenstein series

$$E_{2k}^* := (1 - p^{2k-1})\frac{\zeta(1-2k)}{2} + \sum_{n=1}^{\infty} \sigma_{2k-1}^*(n) q^n \in \mathbb{Q} + q\mathbb{Z}[[q]];$$

$$\text{here and throughout,} \quad \sigma_\alpha^*(n) := \sum_{d|n,\,(d,p)=1} d^\alpha.$$

p-adic Eisenstein series

We have also the non-algebraic p-adic Eisenstein series of the negative (opposite) weight $-2k$:

$$E_{-2k} := \frac{\zeta_p(2k+1)}{2} + \sum_{n=1}^{\infty} \sigma^*_{-2k-1}(n) q^n \in \frac{\zeta_p(2k+1)}{2} + q\mathbb{Q}[[q]]$$
$$=: \frac{\zeta_p(2k+1)}{2} + E'_{-2k}.$$

It is an overconvergent U_p-eigenform of weight $-2k$ and level $\Gamma_0(p)$. Then the product

$$H := E^*_{2k} E_{-2k} = E^*_{2k} \cdot (E'_{-2k} + \frac{\zeta_p(2k+1)}{2}).$$

is a weight 0 overconvergent U_p-eigenform. Its slope is finite (non-zero eigenvalue).

Overconvergent eigenforms of finite slope: Buzzard's analytic continuation theorem

The reference is:

Buzzard K.: Analytic continuation of overconvergent eigenforms, J. Amer. Math. Soc., vol. **16***, no. 1, pp. 29–55.*

Calegari F.: Irrationality of certain p-adic periods for small p, IMRN, no. 20 (2005), pp. 1235–1249.

The statement (in a particular case) is that if f is a rigid-analytic section of $\omega^{\otimes k}$ over a strict neighborhood of the rigid connected component of the ordinary locus containing the cusp $\infty \in X_0(p)$, and which is an eigenform for $U_p f = a_p f$ with a non-zero eigenvalue $a_p \neq 0$, then f has an automatic analytic continuation across the entire supersingular locus (stopping, with a natural boundary unless f is algebraic, at the rigid connected component of the other cusp $0 \in X_0(p)$).

The 2-adic ordinary disks in $X_0(2)$: applying Buzzard's theorem in Calegari's method

Let now $p = 2$. We work on the modular curve $X_0(2) \cong \mathbb{P}^1$ with the Hauptmodul

$$x = x(q) := \frac{\Delta(2\tau)}{\Delta(\tau)} = q \prod_{n=1}^{\infty} (1 + q^n)^{24},$$

in which we may formally expand

$$q = x - 24x^2 + 852x^3 - 35744x^4 + \cdots \in x + x^2\mathbb{Z}[[x]].$$

In this coordinate, it is readily seen that the ordinary component of the cusp ∞ is just the unit disk $|x|_2 \leq 1$. As the Fricke involution w_2 swaps $2^{12}x$ with $1/x$, it follows that the ordinary component of the other cusp 0 is given by $|x|_2 \geq 2^{12}$.

The 2-adic ordinary disks in $X_0(2)$: applying Buzzard's theorem in Calegari's method

Upshot: Buzzard's analytic continuation for the overconvergent weight-0 eigenform $H = E^*_{2k} E_{-2k}$ means precisely that $H(x)$ is convergent on the disk $|x|_2 \leq 2^{12}$. (But its two individual factors $E^*_{2k}(x)$ and $E_{-2k}(x)$ are both only convergent on $|x|_2 \leq 1$.)

Suppose now for contradiction that $\zeta_2(2k+1) \in \mathbb{Q}$; meaning precisely that $H(x) \in \mathbb{Q}[[x]]$. Then

$$\tau(H(x)) = 2k{+}1, \quad \text{convergence radii } \rho_2 = 2^{12}, \rho_p = 1 \text{ at } p \notin \{2, \infty\};$$

...but $\rho_\infty = 2^{-6}$, since $(1+i)/2$ is an elliptic point and one checks $x((1+i)/2) = -2^{-6}$.
We have $3 < \sum_{v \in M_{\mathbb{Q}}} \log \rho_v = 6 \log 2 < 5$, so at this point Calegari in [IMRN, 2005] could conclude irrationality of $\zeta_2(3)$ but not of $\zeta_2(5)$.

Conclusion of the irrationality proof for $\zeta_2(5)$

We can now continue his method by applying the arithmetic holonomicity theorem. Set $k = 2$ in the previous, whence — assuming for contradiction that $\zeta_2(5) \in \mathbb{Q}$ — we have $\tau(H(x)) = 5$ with radii $r_2 = 2^{12}$ and $r_p = 1$ for all odd primes p. For the Archimedean region, follow $x(q)$ by the further fractional-linear transformation

$$q(z) \,:\, \{z \in \mathbb{C} : |z| < 1/5\} \to B := \{q \in \mathbb{C} : |q + 3/16| < 5/16\},$$
$$z \mapsto \frac{z}{1+3z} = z + z^2 \sum_{n=0}^{\infty} (-3)^{n+1} z^n,$$

a conformal isomorphism from the centered disk $D(0; 1/5)$ in the z-plane onto the pointed domain $(B, 0)$ in the q-plane, showing in particular that the latter has conformal mapping radius $1/5$. In effect we use the analyticity of $H(x(q)) \in \mathbb{C}[[q]]$ on the region B in the q-plane.

Conclusion of the irrationality proof for $\zeta_2(5)$

One calculates

$$\sup_{\partial B} |x(q)| = |x(1/8)| = 3.2316\ldots,$$

a reasonably small value (compare to $\sup_{|q|=1/5} |x(q)| = |x(1/5)| = 51.768\ldots$ for the same radius $1/5$ in the q-plane). We thus select:

- $R_p := 1$ and $x_p(z) = z$, if $p \notin \{2, \infty\}$;
- $R_2 := 2^{12}$ at the 2-adic place and $x_2(z) = z$;
- $R_\infty := 1/5$ at the Archimedean place and

$$x_\infty(z) := x(q(z)) = x(z/(1+3z)) = \frac{z}{1+3z} \prod_{n=1}^{\infty} \Big(1 + \big(\frac{z}{1+3z}\big)^n\Big)^{24}.$$

Conclusion of the irrationality proof for $\zeta_2(5)$

With those numerics now, the arithmetic holonomicity theorem yields an upper bound by

$$\leq \frac{12\log 2 + \log^+ |x(1/8)|}{12\log 2 + \log(1/5) - 5} < \frac{9.5}{6.7 - 5} = 5.58\ldots < 6$$

on the minimal order of a linear ODE satisfied by $H(x)$ over $\mathbb{Q}(x)$. This is a contradiction, since it is well-known that $H(x)$ is holonomic with minimal $r = 6$. It is a general fact that for $f(\tau)$ a modular form of weight w, and $x(\tau)$ a non-constant modular function, the multi-valued function $f(x)$ satisfies a linear differential equation with algebraic function coefficients of the minimum order $= w + 1$. (See section 2.3 of Kontsevich and Zagier's paper *Periods*, *in:* Mathematics Unlimited—2001 and Beyond, *Springer* (2001), pp. 771–808.)

The contradiction only means that $\zeta_2(5) \notin \mathbb{Q}$.