Modular Forms: Example Sheet 3

1. Show that for $N \geq 5$ $X(\Gamma_1(N))$ has no elliptic points and every cusp is regular.

2. Suppose n and N are positive integers and let S_n^N be the set of matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = n$, $a \geq 1$ coprime to N and $0 \leq b < d$. If L is a lattice in $\mathbb{C}$ and $t \in \mathbb{C}/L$ is a point of order N, show that L has a basis ω_1, ω_2 with $t = \omega_2/N + L$ and $\omega_1/\omega_2 \in \mathcal{H}$. For $\sigma \in S_n^N$ let L_σ denote the lattice with basis $(\frac{a}{n}\omega_1 + \frac{b}{n}\omega_2, \frac{d}{n}\omega_2)$ (with ω_1, ω_2 as above).

   Finally, show that $\sigma \mapsto L_\sigma$ induces a bijection between S_n^N and lattices $L' \supset L$ with $[L' : L] = n$ such that $t + L'$ has order N in $\mathbb{C}/L'$.

3. Show that, as operators on the q-expansions of modular forms in $M_k(N, \chi)$, there is an identity

$$T_n = \sum_{0<d|n} \chi(d) d^{k-1} V_d \circ U_{n/d}.$$

   Deduce that for $f \in M_k(N, \chi)$ with $f = \sum_n a_n q^n$, we have

$$T_m f = \sum_n b_n q^n$$

   where

$$b_n = \sum_{0<d|\gcd(m,n)} \chi(d) d^{k-1} a_{mn/d^2}.$$

4. Show that for $f \in S_k(\Gamma)$, with $f = \sum_n a_n q^n$, there is a constant C such that $|a_n| \leq C n^{k/2}$ for all n.

5. Let $S_0 \subset S_k(\Gamma_0(N))$ denote the sub-$\mathbb{Q}$-vector space consisting of forms $f = \sum_n a_n q^n$ with $a_n \in \mathbb{Q}$. You may assume that $\dim_{\mathbb{Q}}(S_0) = \dim_{\mathbb{C}} S_k(\Gamma_0(N))$.

   (a) Show that if $f = \sum_n a_n q^n \in S_k(\Gamma_0(N))$ is a normalised eigenform then the extension of $\mathbb{Q}$ generated by all the a_n is a number field.

   (b) Show that if $f = \sum_n a_n q^n \in S_k(\Gamma_0(N))$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, then $f^\sigma := \sum_n \sigma(a_n) q^n \in S_k(\Gamma_0(N))$.

   (c) Deduce that if f is a normalised eigenform then the extension of $\mathbb{Q}$ generated by the T_n eigenvalues of f with n coprime to N is a totally real number field.

6. If $\Gamma' \subset \Gamma$ and $-I \notin \Gamma$ define a map

$$\mathrm{tr}_{\Gamma'/\Gamma} : S_k(\Gamma') \to S_k(\Gamma)$$

   by $f \mapsto \sum_{\gamma \in \Gamma' \backslash \Gamma} f|_{\gamma,k}$.

   (a) Show that for $f \in S_k(\Gamma)$, $g \in S_k(\Gamma')$

$$[\Gamma : \Gamma'] \langle f, g \rangle = \langle f, \mathrm{tr}_{\Gamma'/\Gamma} g \rangle.$$

   Suppose $\Gamma = \Gamma_1(N)$ (for $N > 3$) and set $\Gamma' = \Gamma_1(N) \cap \Gamma_0(p)$ and $\Gamma'' = \Gamma_1(N) \cap \Gamma^0(p)$ with $p \nmid N$ (here $\Gamma^0(p)$ is the congruence subgroup given by matrices which are *lower* triangular mod p).

   Write $S_k(\Gamma')^{p-\mathrm{old}}$ for the subspace of $S_k(\Gamma')$ spanned by $f(\tau)$ and $f(p\tau)$ as f varies over $S_k(\Gamma)$.

(b) Show that for $f \in S_k(\Gamma')$, $f(\tau/p) \in S_k(\Gamma'')$.

(c) Show that the orthogonal complement of $S_k(\Gamma')^{p-\text{old}}$ is equal to the kernel of the map $S_k(\Gamma') \to S_k(\Gamma)^{\oplus 2}$ given by

$$f \mapsto (\mathrm{tr}_{\Gamma'/\Gamma} f(\tau), \mathrm{tr}_{\Gamma''/\Gamma} f(\tau/p)).$$

7. Admit the following fact: if $f = \sum_n a_n q^n \in S_k(\Gamma_1(N))$ with $a_n = 0$ for all n coprime to N, then $f = \sum_{p|N} f_p(p\tau)$ for some $f_p \in S_k(\Gamma_1(N/p))$.

   Show that if $f \in S_k(N,\chi)^{\text{new}}$ is an eigenvector for the operators T_n with n coprime to N, then f is an eigenform (for all the T_n).

   You will need to assume that $S_k(N,\chi)^{\text{new}}$ is stable under the action of all the Hecke operators. If you like, prove this as an extra exercise.

   Recall that a normalised eigenform in $S_k(N,\chi)^{\text{new}}$ is called a *newform*. Show that $S_k(\Gamma_1(N)$ is spanned by the set

$$\{f(d\tau) : f \text{ is a newform of level } M \text{ and } dM \mid N\}.$$

   In fact this set is a basis for $S_k(\Gamma_1(N))$.

8. Let $w_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$ and consider the operator $W : S_k(\Gamma_1(N)) \to S_k(\Gamma_1(N))$ given by $f \mapsto i^k f|_{w_N,k}$. Show that W^2 is the identity, and that W gives an isomorphism between $S_k(N,\chi)$ and $S_k(N,\chi^{-1})$. Furthermore, show that W is self-adjoint, i.e. $\langle Wf, g\rangle = \langle f, Wg\rangle$ for $f, g \in S_k(\Gamma_1(N))$.

9. Show that if two Dirichlet series $F(s) = \sum_{n\geq 1} a_n n^{-s}$, $G(s) = \sum_{n\geq 1} b_n n^{-s}$, converge absolutely in $\mathrm{Re}(s) > \sigma$ for some $\sigma > 0$ and satisfy $F(s) = G(s)$ for $\mathrm{Re}(s) > \sigma$ then $a_n = b_n$ for all n.

Comments, queries and corrections can be sent by email to `jjmn2@cam.ac.uk`.