M3/4/5P12 PROGRESS TEST 1

Note: all representations are assumed to be on finite dimensional complex vector spaces.

Question 1. Let G be a finite group.

(a) What does it mean for a (non-zero) representation of G to be *irreducible*?

(b) Show that if V is an irreducible representation of G and

$$f : V \to V$$

is a G-linear map then f is equal to multiplication by a scalar $\lambda \in \mathbb{C}$. *You may assume the fact that every linear map from V to V has an eigenvalue.*

(c) Using part (b), show that a non-zero irreducible representation of a finite Abelian group is one-dimensional.

(d) Now let $G = C_3 = \{e, g, g^2\}$ the cyclic group of order 3. Consider the regular representation $\mathbb{C}G$. What are the irreducible non-zero subrepresentations of $\mathbb{C}G$?

Solution 1. (a) It means that the only subrepresentations of V are $\{0\}$ and V. **1 mark**

You have to mention that $\{0\}$ and V are allowed as subrepresentations!

(b) f has an eigenvalue λ. So $f - \lambda \operatorname{id}_V$ is a G-linear map from V to V (since g and id_V are both G-linear, $f - \lambda \operatorname{id}_V$ is G-linear), with non-zero kernel (because λ is an eigenvalue). The kernel is a subrepresentation of V (it's fine to just state this). By the definition of an irreducible representation the kernel of $f - \lambda \operatorname{id}_V$ must be $\{0\}$ or V and so it must be V. So $f - \lambda \operatorname{id}_V = 0$ and hence so f is equal to multiplication by λ. **3 marks**

(c) Let G be a finite Abelian group and let V be an irrep of G. For $g \in G$ the linear map $\rho_V(g) : V \to V$ is G-linear, since $\rho_V(h) \circ \rho_V(g) = \rho_V(hg) = \rho_V(gh) = \rho_V(g) \circ \rho_V(h)$ for all $h \in G$. So by part b) $\rho_V(g)$ is equal to multiplication by a scalar $\lambda_g \in \mathbb{C}$ (the scalar can depend on h). Since this is true for all $g \in G$, any one-dimensional subspace of V is G-stable. V is irreducible, so this one-dimensional subrepresentation must be all of V, and hence V is one-dimensional. **3 marks**

(d) We know that the irreducible representations V_0, V_1, V_2 of C_3 are one dimensional, and that the action of g on V_j is multiplication by ω^j, where $\omega = e^{2\pi i/3}$. So we need to find three non-zero vectors v_0, v_1, v_2 in $\mathbb{C}G$ with $\rho_V(g)v_j = \omega^j v_j$. Write $v_j = a_j[e] + b_j[g] + c_j[g^2]$. Then $\rho_V(g)v_j = a_j[g] + b_j[g^2] + c_j[e]$. This implies that $b_j = \omega^j c_j$ and $c_j = \omega^j a_j$. Taking $a_j = 1$ we conclude that the irreducible subrepresentations of $\mathbb{C}G$ are the three one-dimensional subspaces spanned by $v_0 = [e]+[g]+[g^2]$, $v_1 = [e]+\omega^2[g]+\omega[g^2]$ and $v_2 = [e] + \omega[g] + \omega^2[g^2]$. **3 marks**

If you listed the 3 isomorphism classes of irreps of C_3, I gave 2 marks — for full credit I wanted to see *subrepresentations* of $\mathbb{C}C_3$, i.e. explicit G-stable subspaces.

Question 2. Consider the dihedral group of order 8,

$$D_8 = \langle s, t : s^4 = t^2 = e, tst = s^{-1} \rangle.$$

Date: Tuesday 23rd February, 2016.

There is a three-dimensional representation of D_8 on $V = \mathbb{C}^3$ defined by

$$\rho_V(s) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \qquad \rho_V(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

You don't need to check that this defines a representation.

(a) Find a one-dimensional subrepresentation U_1 of V. Can you find another one?

(b) Deduce that V is isomorphic as a representation of D_8 to a direct sum $U_1 \oplus U_2$ where U_2 is a two-dimensional irreducible representation of D_8. *You don't need to find U_2.*

(c) Show that D_8 has one isomorphism class of two-dimensional irreducible representations and four isomorphism classes of one-dimensional representations. You may assume any results you need from the course.

Solution 2. (a) We have to find a simultaneous eigenvector for $\rho_V(s)$ and $\rho_V(t)$. The eigenspaces for $\rho_V(t)$ are $\{(a,b,a) : a,b \in \mathbb{C}\}$ with eigenvalue 1, and $\{(a,0,-a) : a \in \mathbb{C}\}$ with eigenvalue -1. We have $\rho_V(s)(a,b,a) = (a,0,a-b)$ so a simultaneous eigenspace is given by $(a,0,a)$. So let U_1 be the span of $(1,0,1)$. (1 mark if you recognised that we need to find a simultaneous eigenvector, 1 more mark for finding U_1).

There are no other one-dimensional subrepresentations of V. 2 more marks for proving this. Here are two possible arguments:

- Since $\rho_V(s)$ takes (a,b,a) to $(a,0,-a)$ the only eigenvectors for $\rho_V(s)$ in the 1-eigenspace for $\rho_V(t)$ are multiples of $(1,0,1)$. So these are the only (non-zero) simultaneous eigenvectors with $\rho_V(t)$ eigenvalue equal to 1.
  Now we consider the -1-eigenspace for $\rho_V(t)$. These are all scalar multiples of $(1,0,-1)$, and this is not an eigenvector for $\rho_V(s)$, so there are no (non-zero) simultaneous eigenvectors with $\rho_V(t)$ eigenvalue equal to -1. We deduce that U_1 is the only one-dimensional subrepresentation.
- Here's an alternative proof. Suppose there was another one-dimensional subrep of V. Since V is isomorphic to a direct sum of irreducibles, and it has dimension three, if it has two one-dimensional subreps then it has to be a direct sum of *three* one-dimensional representations. This implies that $\rho_V(s)$ and $\rho_V(t)$ are diagonal with respect to some basis, so $\rho_V(s)$ and $\rho_V(t)$ commute. But you can check that $\rho_V(s)\rho_V(t) \neq \rho_V(t)\rho_V(s)$.

Total: **4 marks**

Note that we are looking for a simultaneous eigenvector of $\rho_V(s)$ and $\rho_V(t)$ — this eigenvector does not have to have the same *eigenvalue* for $\rho_V(s)$ and $\rho_V(t)$.

A trick to remember: we know that $\rho_V(t)^2 = \mathrm{id}_V$, so the only possible eigenvalues for $\rho_V(t)$ are 1 and -1. So you don't need to work out the characteristic polynomial, you can just look for eigenvectors with these eigenvalues.

(b) By Maschke's theorem, there is a subrepresentation $U_2 \subset V$ which is complementary to U_1. The dimension of U_2 is equal to 2. (1 mark for applying Maschke's theorem). Since there are no other one-dimensional subrepresentations of V, U_2 is irreducible. (1 mark for explaining why U_2 is irreducible)
Total: **2 marks**

(c) We have shown that there is at least one irreducible two-dimensional rep of D_8. We also know that there's at least one one-dimensional rep of D_8.

We have $8 = \sum_{i=1}^{r} d_i^2$ where d_i denotes the dimensions of the irreps. Since $8 - 2^2 - 1^2 = 3 < 2^2$ the remaining irreps must all be one-dimensional. So the only possibility for the d_i's is four 1s and one 2. **4 marks**