M3/4/5P12 PROBLEM SHEET ON MASTERY MATERIAL

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Exercise 1. Let G be a finite group, with $H \subset G$ a subgroup and let V be a representation of G. Suppose $W \subset \mathrm{Res}_H^G V$ is a subrepresentation of the restriction of V to a representation of H.

(a) Let $g \in G$, and consider the subspace $\rho_V(g)W \subset V$. Show that this subspace depends only on the left coset gH of g.

(b) If $C \in G/H$ is a left coset, write W_C for the subspace $\rho_V(g)W \subset V$, where $g \in C$. Fix a representative g_C for each left coset C and let $f : G \to W$ be an element of $\mathrm{Ind}_H^G W$. Show that

$$\rho_V(g_C)f(g_C^{-1}) \in W_C$$

is independent of the choice of coset representative g_C.

(c) Suppose the subspaces $\{W_C : C \in G/H\}$ together sum to give V and, the sum is direct. In other words, we have

$$V = \bigoplus_{C \in G/H} W_C.$$

Show that V is isomorphic to the induced representation $\mathrm{Ind}_H^G W$.
Hint: consider the map which takes $f \in \mathrm{Ind}_H^G W$ to $\sum_{C \in G/H} \rho_V(g_C)f(g_C^{-1})$.

This exercise shows that our definition of the induced representation gives something satisfying the (alternative) definition given by Serre in *Linear representations of finite groups*.

Exercise 2. Let G be a finite group and suppose we have a subgroup $H \subset G$ and a subgroup $K \subset H$. Let W be a representation of K. Consider the representation

$$IW = \mathrm{Ind}_H^G(\mathrm{Ind}_K^H W).$$

(a) Show that if V is a representation of G, we have

$$\langle \chi_{IW}, \chi_V \rangle = \langle \chi_W, \chi_{\mathrm{Res}_K^G V} \rangle$$

(b) Show, using part a), that IW is isomorphic to $\mathrm{Ind}_K^G W$. *You can also try to show this directly, without using character theory.*

Exercise 3. Let $G = S_5$ and let $H = A_4$ be the subgroup of G given by even permutations of $\{1, 2, 3, 4\}$ which fix 5.

Let V be a three-dimensional irreducible representation of H (there's a unique such V up to isomorphism, see Question 3 on Sheet 4). Use Frobenius reciprocity to compute the decomposition of $\mathrm{Ind}_H^G V$ as a direct sum of irreducible representations of G (you can freely refer to the character table of S_5 — this is computed in Exercise 2 in the 'extra exercises' for Sheet 4).

Exercise 4. Suppose H is a subgroup of a finite group G, and let V be an irreducible representation of H. Let $\chi_1, \dots, \chi_r$ be the irreducible characters of G and suppose that

$$\chi_{\mathrm{Ind}_H^G V} = \sum_{i=1}^{r} d_i \chi_i.$$

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Show that $\sum_{i=1}^{r} d_i^2 \leq [G : H]$.

Exercise 5. Suppose H is a subgroup of a finite group G, and let V be a representation of H. Let $g \in G$ with conjugacy class $C(g)$. Suppose that $C(g) \cap H = D_1 \cup D_2 \cup \cdots \cup D_t$, where the D_i are conjugacy classes in H. Note that we can evaluate the character χ_V of V on each conjugacy class D_i, by defining $\chi_V(D_i) = \chi_V(h)$ for $h \in D_i$.

(a) Show that the character χ of $\mathrm{Ind}_H^G V$ is given by

$$\chi(g) = \frac{|G|}{|H|} \sum_{i=1}^{t} \frac{|D_i|}{|C(g)|} \chi_V(D_i)$$

(b) If V is the trivial one-dimensional representation, show that the character χ of $\mathrm{Ind}_H^G V$ is given by

$$\chi(g) = \frac{|G||C(g) \cap H|}{|H||C(g)|}$$