M3/4/5P12 PROBLEM SHEET 5

Please send any corrections or queries to `j.newton@imperial.ac.uk`. The first exercise is left over from the chapter on character theory.

Exercise 1. Let G, H be two finite groups, let V be a representation of G and let W be a representation of H. Define a natural action of the product group $G \times H$ on the vector space $V \otimes W$ by

$$\rho_{V \otimes W}(g, h)(v \otimes w) = \rho_V(g)v \otimes \rho_W(h)w.$$

This defines a representation of $G \times H$.

(a) Find the character of $V \otimes W$ as a representation of $G \times H$, in terms of the characters χ_V of V and χ_W of W.
(b) Suppose V is an irrep of G and W is an irrep of H. Show that $V \otimes W$ is an irrep of $G \times H$.
(c) Supposes G has r distinct irreducible characters and H has s distinct irreducible characters. Show that $G \times H$ has at least rs distinct irreducible characters. By computing dimensions, show that $G \times H$ has exactly rs distinct irreducible characters and describe them in terms of the irreducible characters of G and of H.

The rest of the exercises are on algebras and modules.

Exercise 2. Find an isomorphism of algebras between $\mathbb{C}[C_3]$ and $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$.

Exercise 3. Let A and B be algebras. Show that the projection map $p : A \oplus B \to A$ defined by $p(a, b) = a$ is an algebra homomorphism, but that the inclusion map $i : A \to A \oplus B$ defined by $i(a) = (a, 0)$ is not.

Exercise 4. Let A and B be algebras. Suppose M is an A-module and N is a B-module. The vector space $M \oplus N$ is naturally an $A \oplus B$-module, with action of $A \oplus B$ given by

$$(a, b) \cdot (m, n) = (a \cdot m, b \cdot n).$$

(a) Let X be an $A \oplus B$-module. Show that multiplication by $e_A := (1_A, 0)$ defines an $A \oplus B$-linear projection map
$$e_A : X \to X.$$
(b) Write $e_A X$ for the image of multiplication by e_A. Show that for $x \in e_A X$ we have $(a, b) \cdot x = (a, 0) \cdot x$ for all $a \in A, b \in B$.
(c) Show that there is an A-module M and a B-module N such that X is isomorphic to $M \oplus N$ as an $A \oplus B$-module.
(d) Describe the simple modules for $A \oplus B$ in terms of the simple modules for A and the simple modules for B.

Exercise 5. Show that the matrix algebra $M_n(\mathbb{C})$ is isomorphic to its own opposite algebra.

Exercise 6. (a) What is the centre of $M_n(\mathbb{C})$?
Hint: $M_n(\mathbb{C})$ has a basis given by matrices E_{ij} with a 1 in the (i, j) entry and 0 everywhere else. Work out what it means for a matrix to commute with E_{ij}.

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(b) If A and B are algebras, show that $Z(A \oplus B) = Z(A) \oplus Z(B)$.
(c) Let $n_1, \ldots, n_r$ be positive integers. What is the centre of the algebra

$$\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})?$$

Exercise 7. Let A be an algebra. Show that the map $f \mapsto f(1_A)$ gives an isomorphism of algebras between $\mathrm{Hom}_A(A, A)$ and A^{op}.

Exercise 8. Let $A = \mathbb{C}[x]/(x^2)$ — recall that this has as a basis $\{1, x\}$, with 1 a unit and $x^2 = 0$. Show that A itself is not a semisimple A-module.