# M3/4/5P12 PROBLEM SHEET 4 (EXTRA EXERCISES)

Please send any corrections or queries to `j.newton@imperial.ac.uk`. These additional exercises work out the character tables of $S_5$ and $A_5$. They are fairly long/tricky but I've included them because it's good to see the computation of these character tables!

**Exercise 1.** Let $G = S_n$ and set $\Omega = \{1, \ldots, n\}$. Recall that we have an $n$-dimensional rep $\mathbb{C}\Omega$ of $S_n$, with a one-dimensional subrepresentation spanned by $\sum_{i=1}^{n}[i]$. Let $V \subset \mathbb{C}\Omega$ be a complementary subrepresentation to this one-dimensional rep. The aim of this exercise is to show that $V$ is irreducible.

For $g \in S_n$ write $Fix_\Omega(g)$ for the subset $\{i \in \Omega : gi = i\} \subset \Omega$. Recall that

$$\chi_{\mathbb{C}\Omega}(g) = |Fix_\Omega(g)|.$$

(See Exercise 7 on Problem Sheet 3).

(1) For $i, j \in \Omega$ define $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$. Show that

$$|Fix_\Omega(g)| = \sum_{i=1}^{n} \delta_{gi,i}.$$

(2) Show that

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = \frac{1}{n!} \sum_{g \in S_n} \left( \sum_{i=1}^{n} \delta_{gi,i} \right)^2.$$

(3) By multiplying out the square in the previous equation, and reordering the sum, show that

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = \frac{1}{n!} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{g \in S_n} \delta_{gi,i}\delta_{gj,j}.$$

(4) Show that if $i = j$ then

$$\sum_{g \in S_n} \delta_{gi,i}\delta_{gj,j} = (n-1)!$$

(5) Show that if $i \neq j$ then

$$\sum_{g \in S_n} \delta_{gi,i}\delta_{gj,j} = (n-2)!$$

(6) Deduce that

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = 2.$$

(7) Finally, show that

$$\langle \chi_V, \chi_V \rangle = 1$$

and deduce that $V$ is an irreducible representation of $S_n$.

**Exercise 2.** There are 7 conjugacy classes in $S_5$, with representatives

$$e, (12), (123), (1234), (12345), (12)(34), (12)(345)$$

and sizes

$$1, 10, 20, 30, 24, 15, 20$$

respectively.

Recall that the one-dimensional characters of $S_5$ are given by $\chi_{triv}$ and $\chi_{sign}$.

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(1) In this previous exercise we found a four-dimensional irrep $V$ for $S_5$. Write down the character $\chi_V$ of $V$ and show that $V' := V \otimes V_{sign}$ gives a four-dimensional irrep which is not isomorphic to $V$.
(2) Using Exercise 6 on Problem Sheet 3, find the character of $\wedge^2 V$ and show that $\wedge^2 V$ is irreducible.
(3) Again using Exercise 6 on Problem Sheet 3, find the character of $S^2V$ and show that $S^2 V \cong V_{triv} \oplus V \oplus W$, where $W$ is a representation of dimension 5. Show moreover that $W$ is irreducible, and $W' := W \otimes V_{sign}$ is another, non-isomorphic, irrep of dimension 5.

We have now found all the irreps of $S_5$, and their characters. There are 7 isomorphism classes of irreps: $V_{triv}, V_{sign}, V, V', \wedge^2 V, W$ and $W'$.

**Exercise 3.** There are 5 conjugacy classes in $A_5$, with representatives

$$e, (123), (12345), (13452), (12)(34)$$

and sizes

$$1, 20, 12, 12, 15$$

respectively.

(1) Show that the representations $V$ and $W$ of the previous exercise restrict to irreducible representations of $A_5$ (which we still call $V, W$).
(2) Show that the representation $\wedge^2 V$ restricts to a representation $X$ of $A_5$ whose character $\chi_X$ has $\langle \chi_X, \chi_X \rangle = 2$. Deduce that $X$ decomposes as a direct sum of two non-isomorphic irreducible representations $Y$, $Z$ of $A_5$.
(3) Deduce that the complete list of irreps (up to isomorphism) of $A_5$ is given by $V_{triv}, V, W, Y, Z$, and show that $\dim(Y)^2 + \dim(Z)^2 = 18$, hence $\dim(Y) = \dim(Z) = 3$.

Here's the character table so far (note that we know $\chi_Y + \chi_Z$ because we know $\chi_X$):

| | $e$ | (123) | (12345) | (13452) | (12)(34) |
|---|---|---|---|---|---|
| $\chi_{triv}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_V$ | 4 | 1 | -1 | -1 | 0 |
| $\chi_W$ | 5 | -1 | 0 | 0 | 1 |
| $\chi_Y$ | 3 | $a$ | $b$ | $c$ | $d$ |
| $\chi_Z$ | 3 | $-a$ | $1-b$ | $1-c$ | $-2-d$ |

(4) Show that if $V$ is a rep of $A_5$ then $\overline{\chi_V(g)} = \chi_V(g)$. *Hint: If $g \in A_5$ then $g^{-1}$ is conjugate to $g$, so $\chi_V(g^{-1}) = \chi_V(g)$.*
(5) Using the column orthogonality relations

$$\sum_{i=1}^{r} |\chi_i(g)|^2 = |G|/|C(g)|$$

where $C(g)$ is the conjugacy class of $g$, show that $a = 0$, $d = -1$ and $b, c$ are both solutions to the quadratic equation $x^2 - x - 1 = 0$.
(6) Conclude that the character table of $A_5$ is given by

| | $e$ | (123) | (12345) | (13452) | (12)(34) |
|---|---|---|---|---|---|
| $\chi_{triv}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_V$ | 4 | 1 | -1 | -1 | 0 |
| $\chi_W$ | 5 | -1 | 0 | 0 | 1 |
| $\chi_Y$ | 3 | 0 | $\frac{1+\sqrt{5}}{2}$ | $\frac{1-\sqrt{5}}{2}$ | -1 |
| $\chi_Z$ | 3 | 0 | $\frac{1-\sqrt{5}}{2}$ | $\frac{1+\sqrt{5}}{2}$ | -1 |