M3/4/5P12 PROBLEM SHEET 2

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Exercise 1. Let V, ρ_V and W, ρ_W be representations of a group G with dimension m and n respectively. Let $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$ be bases for V and W.

Let $A \oplus B$ be the basis for $V \oplus W$ given by $(a_1, 0), \dots, (a_m, 0), (0, b_1), \dots, (0, b_n)$. Describe the matrix representation

$$(\rho_V \oplus \rho_W)_{A \oplus B} : G \to \mathrm{GL}_{m+n}(\mathbb{C})$$

in terms of the matrix representations $(\rho_V)_A$ and $(\rho_W)_B$.

Exercise 2. Let V and W be representations of a group G. Recall that $\mathrm{Hom}_{\mathbb{C}}(V, W)$ denotes the complex vector space of linear maps from V to W.

(1) Let $g \in G$ act on $\mathrm{Hom}_{\mathbb{C}}(V, W)$ by taking a linear map $f : V \to W$ to the linear map
$$g \cdot f : v \mapsto \rho_W(g) f(\rho_V(g^{-1})v).$$
Show that this defines a representation of G on $\mathrm{Hom}_{\mathbb{C}}(V, W)$. What is the dimension of this representation, in terms of the dimensions of V and W?

(2) Show that the invariants $\mathrm{Hom}_{\mathbb{C}}(V, W)^G$ in this representation of G are the G-linear maps $\mathrm{Hom}_G(V, W)$.

Exercise 3. Recall that we proved in lectures that if U is a subrepresentation of a representation V of a finite group, then there exists a complementary subrepresentation $W \subset V$ with $V \cong U \oplus W$ (Maschke's theorem).

Prove by induction that if V is a representation of a finite group G, then V is isomorphic to a direct sum
$$V_1 \oplus V_2 \oplus \cdots \oplus V_d$$
with each V_i an irreducible representation of G. *A proof is written in the typed lecture notes if you get stuck!.*

Exercise 4. (1) Let G be a group and
$$\chi : G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$$
a group homomorphism (i.e. a one-dimensional matrix representation). Show that if $g, h \in G$ then $\chi(g) = \chi(hgh^{-1})$.

(2) We let $G = S_n$. Let $2 \le j \le n$ be an integer. Show that there is an element $h \in S_n$ such that $h(12)h^{-1} = (1j)$. Show moreover that if $g \in S_n$ is any transposition (i.e. $g = (jk)$ for $j \neq k$) then there exists an $h \in G$ such that $h(12)h^{-1} = g$.

(3) Show that there are only two one-dimensional representations of S_n (up to isomorphism), given by the trivial map $S_n \to \{1\}$ and the sign homomorphism $S_n \to \{\pm 1\}$. *Recall that every element of S_n is a product of transpositions.*

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Exercise 5. (1) Let G be a finite group. Write $Z(G)$ for the centre of the group:
$$Z(G) = \{z \in G : zg = gz \forall g \in G\}.$$
Note that $Z(G)$ is a subgroup of G. Let V be an irreducible representation of G. Show that for each $z \in Z(G)$ there exists $\lambda_z \in \mathbb{C}$ such that
$$\rho_V(z)v = \lambda_z v$$
for all $v \in V$.

(2) Suppose V is a faithful irreducible representation of G. Show that $Z(G)$ is a cyclic group. *Hint: A finite subgroup of* $\mathbb{C}^\times$ *is cyclic.*

Exercise 6. (1) Let D_{2n} be the dihedral group of order $2n$, generated by a rotation s of order n and a reflection t of order 2. Recall that we have $tst = s^{-1}$. Let $\zeta \in \mathbb{C}$ be an nth root of unity and let V_ζ be the representation of D_{2n} on the vector space $\mathbb{C}^2$ (with the standard basis) given by
$$\rho_{V_\zeta}(s) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad \rho_{V_\zeta}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Verify that this defines a representation of D_{2n} on V.
Show that if $\zeta \neq \pm 1$ then this representation is irreducible.

(2) What are the one-dimensional matrix representations of D_{2n}?

(3) Show that if n is even, there are $(n+6)/2$ isomorphism classes of irreducible representations of D_{2n}: 4 of dimension one and $(n-2)/2$ of dimension 2.

(4) Show that if n is odd, there are $(n+3)/2$ isomorphism classes of irreducible representations of D_{2n}: 2 of dimension one and $(n-1)/2$ of dimension 2.