M3/4/5P12 PROBLEM SHEET 1

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Exercise 1. (1) Let $G = C_4 \times C_2 = \langle s, t : s^4 = t^2 = e, st = ts \rangle$. Let $V = \mathbb{C}^2$ with the standard basis. Consider the linear transformations of V defined by the matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Verify that sending s to S and t to T defines a representation of G on V. Is this representation faithful?

(2) Now let

$$Q = \begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} -1 & 0 \\ i+1 & 1 \end{pmatrix}.$$

Verify that sending s to Q and t to R also defines a representation of G on V. Is this representation faithful?

(3) Show that S is conjugate to Q and T is conjugate to R. Are the two representations we have defined isomorphic?

Exercise 2. (1) Let G be a finite group, and (V, ρ_V) a representation of G, with V a finite dimensional complex vector space. Let g be an element of G. Show that there is a positive integer $n \geq 1$ such that $\rho_V(g)^n = \mathrm{id}_V$. What can you conclude about the minimal polynomial of $\rho_V(g)$?

(2) Show that $\rho_V(g)$ is diagonalisable.

Exercise 3. (1) Consider S_3 acting on $\Omega = \{1, 2, 3\}$ and write V for the associated permutation representation $\mathbb{C}\Omega$. Write down the matrices giving the action of $(123), (23)$ with respect to the standard basis $([1], [2], [3])$ of V.

(2) Write U for the subspace of V consisting of vectors $\{\lambda_1[1] + \lambda_2[2] + \lambda_3[3] : \lambda_1 + \lambda_2 + \lambda_3 = 0\}$. Show that U is mapped to itself by the action of S_3. Find a basis of U with respect to which the action of (23) is given by a diagonal matrix and write down the matrix giving the action of (123) with respect to this basis.

Can you find a basis of U with respect to which the actions of both (23) and (123) are given by diagonal matrices?

Exercise 4. (1) Let V, W be two representations of G and $f : V \to W$ an invertible G-linear map. Show that f^{-1} is G-linear.

(2) Show that a composition of two G-linear maps is G-linear.

(3) Deduce that 'being isomorphic' is an equivalence relation on representations of a group G.

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Exercise 5. (1) Let G, H be two finite groups, and let $f : G \to H$ be a group homomorphism. Suppose we have a representation V of H. Show that $\rho_V \circ f : G \to \mathrm{GL}(V)$ defines a representation of G. We call this representation the *restriction* of V from H to G along f, written $\mathrm{Res}_f(V)$.

(2) Let S_n act on the set of cosets $\Omega = \{eA_n, (12)A_n\}$ for the alternating group $A_n \subset S_n$ by left multiplication. We get a two-dimensional representation $\mathbb{C}\Omega$ of S_n. Show that $\mathbb{C}\Omega$ is isomorphic to $\mathrm{Res}_{sgn}(V)$ where $sgn : S_n \to \{\pm 1\}$ is the sign homomorphism[1] and V is the regular representation of $\{\pm 1\}$.

Exercise 6. (1) Let $C_n = \langle g : g^n = e \rangle$ be a cyclic group of order n. Let V_{reg} be the regular representation of C_n. What is the matrix for the action of g on V_{reg}, with respect to the basis $[e], [g], \ldots [g^{n-1}]$? What are the eigenvalues of this matrix?

(2) Find a basis for V_{reg} consisting of eigenvectors for $\rho_{V_{reg}}(g)$.

(3) Let G be a finite Abelian group, and let V be a representation of G. Show that V has a basis consisting of simultaneous eigenvectors for the linear maps $\{\rho_V(g) : g \in G\}$. *Hint: recall the fact from linear algebra that a commuting family of diagonalisable linear operators is simultaneously diagonalisable.*

[1] taking even permutations to $+1$ and odd permutations to -1