M3/4/5P12 GROUP REPRESENTATION THEORY

JAMES NEWTON

Course Arrangements

Lecture 1

Send comments, questions, requests etc. to `j.newton@imperial.ac.uk`. The course homepage is `http://wwwf.imperial.ac.uk/~jjmn07/M3P12.html`. Problem sheets and solutions, lecture notes and other information will all be posted here. There is no blackboard page for this course.

Problems classes. Problems classes will take place in the Tuesday lecture slot of odd weeks from week 3 (January 26) onwards.

Assessment. 90% of your mark is from the final exam, 10% from two progress tests (5% each) on Tuesday February 16 and Tuesday March 15.

Office hour. There will an office hour, in office 656, every Thursday 4–5pm, starting week 2 (January 21) and ending week 10 (March 17).

Other reading. The course webpages for 2014 `http://wwwf.imperial.ac.uk/~epsegal/repthy.html` and 2015 `https://sites.google.com/site/matthewtowers/m3p12` contain lecture notes and problem sheets. The course content this year will be very similar, but the material will be reorganised a little.

Here are some textbooks:

- G. James and M. Liebeck, Representations and Characters of Groups. The first 19 sections of this book cover almost all of the course, apart from the final part on semisimple algebras. You should be able to access an ebook version through the Imperial library website.
- J. P. Serre, Linear Representations of Finite Groups. Part I of this book gives a concise and elegant exposition of character theory.
- J. L. Alperin, Local Representation Theory. The first couple of sections cover the part of the course on semisimple algebras. The rest of the book is about group representation theory in positive characteristic, which we don't cover.

Prerequisites. Group theory and linear algebra (as covered in, for example, the second year course Algebra 2).

Date: May 1, 2016.

Course outline. This course is an introduction to the representation theory of finite groups over the complex numbers.

(1) Basic definitions and examples.
(2) 'Structure theory of representations': Maschke's theorem on complete reducibility, Schur's lemma. Duals and tensor products of representations.
(3) Character theory: the character of a group representation, character tables and orthogonality relations.
(4) The group algebra and representations as modules for the group algebra. Semisimple algebras, matrix algebras, the Artin–Wedderburn theorem.

Some vague motivation. Groups often come with interesting actions on sets or vectors spaces. In fact, most groups 'in nature' arise as symmetry groups of mathematical or physical systems. For example, symmetry groups of regular polygons (dihedral groups), symmetric groups, matrix groups ($\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{R})$,...).

Let's recall what a group action on a set is. Suppose we have a (left) group action on a finite set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$. Recall that this means there's a map

$$G \times \Omega \to \Omega$$

written $(g, \omega) \mapsto g \cdot \omega$ such that $e \cdot \omega = \omega$ for all $\omega \in \Omega$ and $g \cdot (h \cdot \omega) = (gh) \cdot \omega$ for all $g, h \in G$ and $\omega \in \Omega$.

I drew the example of D_8, the symmetry group of the square. We can consider D_8 acting on (for example) the set of vertices of the square, or on the whole plane in which the square is sitting.

Studying these group actions is interesting in its own right, and can also be used to study the group itself. For example:

Theorem. *Suppose a group G has cardinality $|G| = p^a q^b$ with p, q prime and $a, b \in \mathbb{Z}_{\geq 0}$. Then G is not simple. In other words, G has a proper normal subgroup.*

The easiest proof of this theorem crucially uses representation theory. We probably won't be able to include it in this course (it uses a small amount of algebraic number theory), but you can find the proof (and the necessary extra material) in James and Liebeck's textbook.

Representation theory also has a wide range of applications across mathematics, physics, chemistry,.... Some examples:

- Particle physics is intimately connected with representation theory, particularly the representation theory of Lie groups and Lie algebras. Fundamental particles like quarks correspond to vectors in representations of symmetry groups (e.g. SU(3)). See `http://arxiv.org/abs/0904.1556` for much more!
- In number theory, if $K/\mathbb{Q}$ is a Galois extension it has a Galois group $\mathrm{Gal}(K/\mathbb{Q})$ and studying representations of $\mathrm{Gal}(K/\mathbb{Q})$ (and related groups) is something of central importance. The proof of Fermat's last theorem crucially relies on the theory of these Galois representations.
- Suppose we have a physical system modelled by (homogeneous, linear) differential equations. The solutions to such a system of differential equations form a vector

space, and if we have a group of symmetries G of our physical system, then G acts on the solution space, making it into a representation of G.

1. Representations

Lecture 2

1.1. First definitions.

Definition 1.1. Let G be a group. A representation of G on a vector space V is a collection of linear maps

$$\rho_V(g) : V \to V$$

for each $g \in G$, such that

(1) $\rho_V(e) = \mathrm{id}_V$
(2) $\rho_V(gh) = \rho_V(g) \circ \rho_V(h)$

If V has dimension d, we say that the representation has dimension d.

Let's give our first examples of group representations:

Example 1.1. (1) Let G be any group, V any vector space, and $\rho(g) = \mathrm{id}_V$ for all $g \in G$. This is called the trivial representation of G on V. If V isn't specified, then 'the trivial representation of G' usually means the trivial representation of G on the one–dimensional vector space $\mathbb{C}$.

(2) Let $G = D_{2n}$, the symmetry group of a regular n-gon $\mathbb{R}^2$ (centred at the origin). D_{2n} acts by rotations and reflections on $\mathbb{R}^2$ (which are linear maps). This defines a representation of D_{2n} on $\mathbb{R}^2$. Extending linearly, we get a representation of D_{2n} on $\mathbb{R}^2$
(3) $G = C_n = \{e, g, g^2, \dots g^{n-1}\}$ a cyclic group of order n. Let $\zeta \in \mu_n(\mathbb{C})$ be an nth root of unity, $V = \mathbb{C}$. Define ρ_V by setting $\rho_V(g^i)$ to be multiplication by ζ^i.
(4) Let G be any group and $\chi : G \to \mathbb{C}^\times$ a group homomorphism. Let V be any complex vector space. We get a representation of G on V by setting

$$\rho_V(g)v = \chi(g)v$$

for $g \in G$ and $v \in V$.
(5) Finally we give an important family of examples. Let $\Omega = \{\omega_1, \dots, \omega_n\}$ be a finite set with an action of G. We define $\mathbb{C}\Omega$ to be the $\mathbb{C}$-vector space with basis $[\omega_1], \dots, [\omega_n]$, so the elements of $\mathbb{C}\Omega$ are formal sums $\sum_{i=1}^n a_i[\omega_i]$ with $a_i \in \mathbb{C}$. For $g \in G$ we define $\rho_{\mathbb{C}\Omega}(g)$ to be the linear map which satisfies

$$\rho_{\mathbb{C}\Omega}(g)[\omega_i] = [g \cdot \omega_i]$$

for $i = 1, \dots n$.

Here are two important special cases of this example:

(a) $G = S_n$, $\Omega = \{1, \dots, n\}$ with the usual permutation action of S_n. This gives an n-dimensional representation of S_n.
(b) G any finite group, $\Omega = G$, with the action given by multiplication of G on itself. So $g \cdot h = gh$ for $g \in G$, $h \in \Omega$. This gives a $|G|$-dimensional representation $\mathbb{C}G$ of G, called the *regular representation*.

Let's give a slightly cleaner definition of a representation.

Definition 1.2. Let V be a vector space. Then $\mathrm{GL}(V)$ is the group whose elements are invertible linear maps $V \to V$, with group multiplication given by composition of maps.

Lemma 1.1. *Let $V, \{\rho_V(g) : g \in G\}$ be a representation of G on V. Then $g \mapsto \rho_V(g)$ gives a homomorphism*

$$\rho_V : G \to \mathrm{GL}(V).$$

Conversely, if $\rho_V : G \to \mathrm{GL}(V)$ is a homomorphism then we get a representation of G on V by letting g acts as $\rho_V(g)$.

Proof. This is immediate from the definitions. □

Now we can rewrite Definition 1.1:

Definition 1.1'. A representation of G on a vector space V is a homomorphism

$$\rho_V : G \to \mathrm{GL}(V).$$

Definition 1.3. A representation is faithful if ρ_V is injective.

Exercise 1.1. For what choices of ζ does Example 3 above give a faithful representation.

Suppose a group G is generated by elements $g_1, \dots g_n$. So every element of G can be written as a product of (powers of) g_i's. In particular, to give a homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

it suffices to give $\rho(g_i)$ for $i = 1, \dots n$.

Example 1.2. Let $G = D_{2n}$. This group has generators s, t with s rotation by $2\pi/n$ and t a reflection. They satisfy relations $s^n = t^2 = e$ and $tst = s^{-1}$. To write down a representation $\rho_V : D_{2n} \to \mathrm{GL}(V)$ it is equivalent to give $S, T \in \mathrm{GL}(V)$ with $S = \rho_V(s)$, $T = \rho_V(t)$, such that $S^n = T^2 = \mathrm{id}_V$ and $TST = S^{-1}$.

For example, let $D = D_8$ and consider Example 2 above. This representation can be
Lecture 3 given explicitly by setting

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

1.2. **Matrix representations.** In the D_8 example we saw that we could describe a group representation by writing down matrices. More generally, to give a map $\rho : G \to \mathrm{GL}(V)$ we can fix a basis $(b_1, \dots, b_n)$ for V and then write down invertible $n \times n$ matrices for each $g \in G$.

Recall that $GL_n(\mathbb{C})$ is the group of invertible $n \times n$ complex matrices. Fixing a basis $B = (b_1, \dots, b_n)$ for V gives an isomorphism

$$P_B : \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C}).$$

We take $f \in \mathrm{GL}(V)$ and pass to the associated matrix $[f]_B \in \mathrm{GL}_n(\mathbb{C})$.

Definition 1.4. A (complex) matrix representation of dimension n of a group G is a homomorphism
$$r : G \to \mathrm{GL}_n(\mathbb{C}).$$

So if (V, ρ) is an n-dimensional representation, and B is a basis of V, we get a homomorphism $\rho_B := P_B \circ \rho : G \to \mathrm{GL}_n(\mathbb{C})$ which is an n-dimensional matrix representation. We're just taking the matrix of each linear map $\rho(g)$ with respect to the basis B.

Conversely, if $r : G \to \mathrm{GL}_n(\mathbb{C})$ is a matrix representation and $V = \mathbb{C}^n$ (thought of as column vectors), then we get $\rho_r : G \to \mathrm{GL}(V)$ by setting $\rho(g)v = r(g)v$, where on the right hand side we do matrix multiplication.

1.3. **Maps of representations, equivalence of matrix representations.** In linear algebra it's important to consider linear maps between different vector spaces. If V and W are two vector spaces then a linear map $f : V \to W$ is a map of sets such that $f(\lambda v+\mu w) = \lambda f(v)+\mu f(w)$. In other words, f 'commutes with' the vector space operations of scalar multiplication and addition. If we have a pair of representations $(V, \rho), (W, \sigma)$ we want to consider linear maps $f : V \to W$ which commute with the action of G.

Definition 1.5. Let (V, ρ_V), (W, ρ_W) be representations of G. A linear map $f : V \to W$ is G-linear if $f(\rho_V(g)v) = \rho_W(g)f(v)$ for all $g \in G$ and $v \in V$.

Definition 1.6. Two representations (V, ρ_V), (W, ρ_W) are isomorphic if there is an invertible G-linear map $f : V \to W$. We say that such an f is a G-linear isomorphism.

Exercise 1.2. Suppose $f : V \to W$ is a G-linear isomorphism. Show that the map $f^{-1} : W \to V$ is G-linear. Deduce that 'being isomorphic' is an equivalence relation on representations.

Proposition 1.1. *Let (V, ρ) and (W, σ) be two representations of dimension m, n respectively. Let $A = a_1, \ldots, a_m$ be a basis for V and $B = b_1, \ldots, b_n$ be a basis for W. Let f be a linear map $f : V \to W$. Let $[f]_{A,B}$ be the $n \times m$ matrix (n rows, m columns) with entries $(f_{ij})_{i,j}$ satisfying $fa_j = \sum_{i=1}^{n} f_{ij}b_i$. Then f is G-linear if and only if $[f]_{A,B}\rho_A(g) = \sigma_B(g)[f]_{A,B}$ for all $g \in G$.*

Proof. Fix $g \in G$. It suffices to check that $f \circ \rho(g) = \sigma(g) \circ f$ if and only if $[f]_{A,B}\rho_A(g) = \sigma_B(g)[f]_{A,B}$. The linear map $f \circ \rho(g) : V \to W$ has associated matrix $[f]_{A,B}\rho_A(g)$ with respect to the bases A, B [matrix multiplication corresponds to composition of linear maps] and $\sigma(g) \circ f$ has associated matrix $\sigma_B(g)[f]_{A,B}$. So we just use the fact that two linear maps are equal if and only if their associated matrices are equal. □

Corollary 1.1. *Let (V, ρ) and (W, σ) be two representations of dimension n. Let $A = a_1, \ldots, a_n$ be a basis for V and $B = b_1, \ldots, b_n$ a basis for W. Then ρ and σ are isomorphic if and only if there is an $n \times n$ invertible matrix P with $P\rho_A(g)P^{-1} = \sigma_B(g)$ for all $g \in G$.*

Proof. First we suppose there is a matrix P as in the statement of the proposition. Let $f : V \to W$ be the linear map with associated matrix $[f]_{A,B} = P$. Since P is invertible, f is invertible. Applying Proposition 1.1 we deduce that f is G-linear. So f gives an isomorphism between ρ and σ.

Second, we suppose that ρ and σ are isomorphic. Let $f : (V, \rho) \to (W, \sigma)$ be an isomorphism, and set $P = [f]_{A,B}$. Applying Proposition 1.1 again, we deduce that $P\rho_A(g) = \sigma_B(g)P$ for all $g \in G$. Rearranging gives $P\rho_A(g)P^{-1} = \sigma_B(g)$ as desired. □

Lecture 4

Definition 1.7. We say that two n-dimensional matrix representations r, s are equivalent if there is an $n \times n$ invertible matrix P with $Pr(g)P^{-1} = s(g)$ for all $g \in G$.

Remark 1.1. Let $(V, \rho), (W, \sigma)$ and A, B be as in Corollary 1.1. The two representations ρ, σ are isomorphic if and only if their associated matrix representations ρ_A, σ_B are equivalent.

Remark 1.2. Let (V, ρ) be a representation of dimension n. Let A and B be two bases for V. Since a representation is isomorphic to itself (via the identity map) Corollary 1.1 says that there is an invertible matrix P such that $P\rho_A(g)P^{-1} = \rho_B(g)$ for all $g \in G$. In fact, we can take P to be the change of basis matrix with entries P_{ij} satisfying $a_j = \sum_{i=1}^n P_{ij} b_i$.

1.4. **Direct sums of representations, indecomposable representations.** One way to build representations of a group is to combine representations together. The simplest way to do this is the *direct sum*. Recall that if V and W are two vector spaces, we get a new vector space $V \oplus W$ whose elements consist of ordered pairs (v, w) with $v \in V$ and $w \in W$. If we have representations (V, ρ), (W, σ) of G there is a natural way to define a representation of G on the vector space $V \oplus W$.

Definition 1.8. Suppose we have linear maps $\alpha : V \to V$ and $\beta : W \to W$. Then we get a linear map $\alpha \oplus \beta : V \oplus W \to V \oplus W$ by defining $(\alpha \oplus \beta)(v, w) = (\alpha(v), \beta(w))$.

Definition 1.9. Let (V, ρ), (W, σ) be representations. The direct sum representation $(V \oplus W, \rho \oplus \sigma)$ is given by

$$\begin{aligned} \rho \oplus \sigma : G &\to \mathrm{GL}(V \oplus W) \\ g &\mapsto \rho(g) \oplus \sigma(g). \end{aligned}$$

Remark 1.3. The injective linear maps $i_V : V \to V \oplus W$ and $i_W : W \to V \oplus W$ given by $v \mapsto (v, 0)$ and $w \mapsto (0, w)$ are G-linear.

Exercise 1.3. Let (V, ρ), (W, σ) be representations. Fix bases A for V and B for W. Write $A \oplus B$ for the basis of $V \oplus W$ given by $(a_1, 0), \ldots, (a_m, 0), (0, b_1), \ldots, (0, b_n)$. Describe the matrix representation $(\rho \oplus \sigma)_{A \oplus B}$ in terms of the matrix representations ρ_A and σ_B.

Definition 1.10. A representation is decomposable if it is isomorphic to a direct sum of two representations of smaller dimension. In other words, we have $V \cong U \oplus W$ with both U and W non-zero. If a representation is not decomposable, we say that it is indecomposable.

Example 1.3. Let's consider the cyclic group with two elements $C_2 = \langle g : g^2 = e \rangle$. What are the one-dimensional representations of C_2? Let V_+ be a one-dimensional vector space with basis vector v_+ and trivial action of C_2. Let V_- be a one-dimensional vector space with basis vector v_- and action of C_2 given by $gv_- = -v_-$. Every one-dimensional representation of C_2 is isomorphic to either V_+ or V_-. We get a two-dimensional representation of C_2, $V_+ \oplus V_-$.

We know another two-dimensional representation of C_2, the regular representation $\mathbb{C}C_2$. This has basis $[e], [g]$ with action of G given by $g[e] = [g]$, $g[g] = [e]$.

Proposition 1.2. *The regular representation $\mathbb{C}C_2$ is isomorphic to $V_+ \oplus V_-$.*

Proof. The map $f : \mathbb{C}C_2 \to V_+ \oplus V_-$ given by $f([e]) = (v_+, v_-)$, $f([g]) = (v_+, -v_-)$ is a G-linear isomorphism. □

Remark 1.4. Let's redo the above proof in terms of matrix representations. The matrix of g under the regular representation with respect to the basis $[e], [g]$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the matrix of g acting on $V_+ \oplus V_-$ with respect to the basis $(v_+, 0), (0, v_-)$ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so to show the two representations are isomorphic we need to find an invertible 2×2 matrix P such that

$$P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In other words we need to diagonalise a matrix. $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ works, and corresponds to the map f in the proof above.

Fact 1.1. Let G be a finite Abelian group and V a representation of G. Then V is isomorphic to a direct sum of one-dimensional representations of G. This follows from Problem 6 on Problem Sheet 1, and amounts to the fact that we can simultaneously diagonalise commuting diagonalisable matrices.

Lecture 5

1.5. Subrepresentations, irreducible representations.

Definition 1.11. A subrepresentation of a representation (V, ρ) for a group G is a vector subspace $W \subset V$ such that $\rho(g)w \in W$ for all $g \in G$ and $w \in W$.

Exercise 1.4. Suppose $g_1, \dots g_r$ generate G and let V be a representation of G. Suppose $W \subset V$ is a subspace such that $\rho(g_i)w \in W$ for all w in W and $i = 1, \dots, r$. Then W is a subrepresentation of V.

Suppose W is a subrepresentation of V. Since $\rho(g)$ restricts to a linear map from W to W for each g, we get a representation $(W, \rho|_W)$. The inclusion map $W \to V$ is obviously G-linear, so we get a morphism of representations $(W, \rho|_W) \to (V, \rho)$.

Example 1.4. Suppose $f : V \to W$ is a G-linear map between representations of G. The kernel of f

$$\ker(f) = \{v \in V : f(v) = 0\}$$

is a subrepresentation of V. The image of f

$$\operatorname{im}(f) = \{f(v) : v \in V\}$$

is a subrepresentation of W.

Example 1.5. Let (V, ρ) be a representation of G. A non-zero simultaneous eigenvector for the linear maps $\{\rho(g) : g \in G\}$ spans a one-dimensional subrepresentation of V. Conversely, if we have a one-dimensional subrepresentation W of V then any basis vector of W is a simultaneous eigenvector for the linear maps $\{\rho(g) : g \in G\}$.

Example 1.6. Let G be any finite group. Let Ω be a finite set with a G-action, and consider the representation $\mathbb{C}\Omega$. The one-dimensional subspace of $\mathbb{C}\Omega$ with basis vector $v_0 = \sum_{\omega\in\Omega}[\omega]$ is a subrepresentation, isomorphic to the trivial representation. Indeed, for every $g \in G$ we have

$$g \cdot v_0 = \sum_{\omega\in\Omega} g[\omega] = \sum_{\omega\in\Omega} [g \cdot \omega] = \sum_{\omega\in\Omega} [\omega].$$

Definition 1.12. If V is a non-zero representation whose only subrepresentations are $\{0\} \subset V$ and $V \subset V$, we say that V is irreducible. If V is non-zero and not irreducible, we say that it is reducible.

Lemma 1.2. *An irreducible representation is indecomposable.*

Proof. Suppose for a contradiction that V is irreducible but decomposable. This means that $V \cong W_1 \oplus W_2$, with both W_1 and W_2 non-zero. But then the image of the map $W_1 \to W_1 \oplus W_2 \to V$ is a proper subrepresentation of V, so V is reducible, which is a contradiction. □

In the next section we will show that the converse holds: an indecomposable representation is always irreducible.

Example 1.7. We saw earlier that the only indecomposable representations of Abelian groups are one-dimensional. Let's give an example of an irreducible two-dimensional representation of S_3. Generators for S_3 are s, t with $s = (123)$ and $t = (23)$. The relations are $s^3 = e, t^2 = e$ and $tst = s^{-1}$. Define a matrix representation by $r(s) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$, $r(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Checking the relations shows that this defines a homomorphism $S_3 \to \mathrm{GL}_2(\mathbb{C})$ and so we get a representation ρ of S_3 on $\mathbb{C}^2$. In fact this is the representation we get by viewing S_3 as the symmetry group of a triangle in $\mathbb{R}^2$.

We're going to show that it's irreducible. Since ρ is two-dimensional, it is reducible if and only if it has a one-dimensional subrepresentation. Equivalently, ρ is reducible if and only if the matrices $r(s)$ and $r(t)$ have a non-zero simultaneous eigenvector. It's an exercise to check that there are no non-zero simultaneous eigenvectors for these matrices.

1.6. **Summary of this chapter.** We defined representations and matrix representations, and gave some examples. We saw how to go between representations and matrix representations by choosing a basis. We defined morphisms of representations (a.k.a G-linear maps), isomorphisms, direct sums and subrepresentations. We defined the notions of indecomposable and irreducible representations and gave an example of a two-dimensional irreducible representation (for S_3).

2. More on representations

2.1. **Maschke's theorem.** The goal of this subsection is to show that every representation of a group G is isomorphic to a finite direct sum of irreducible representations. First we want to understand what it means for a representation to be isomorphic to a direct sum of two representations.

Definition 2.1. Let V be a vector space, and $U \subset V$ a vector subspace. We say that a vector subspace $W \subset V$ is *complementary* to U if $U \cap W = \{0\}$ and $U + W = V$ (i.e. U and W span V).

Proposition 2.1. *Let (V, ρ) be a representation and let $U \subset V$ be a subrepresentation. Suppose $W \subset V$ is a subrepresentation of V, which is complementary to U. Then V is isomorphic to the direct sum representation $U \oplus W$. A G-linear isomorphism is given by $(u, w) \mapsto u + w$.*

Proof. The map $(u, w) \mapsto u + w$ is a G-linear map from $U \oplus W$ to V. We just need to check that it is an invertible map of vector spaces (in other words, an isomorphism of vector spaces). This is now just linear algebra: if $u + w = 0$ then $u = -w \in U \cap W$ and since W is complementary to U we have $u = w = 0$. So the map is injective. It is surjective because $U + W = V$. □

Now we can state the main result of this section of the course:

Theorem 2.1 (Maschke's theorem)**.** *Let G be a finite group, and let V be a representation of G. Suppose we have a subrepresentation $U \subset V$. Then there exists a subrepresentation $W \subset V$ which is complementary to U. We therefore have an isomorphism of representations $V \cong U \oplus W$.*

Remark 2.1.

1. The fact that $V \cong U \oplus W$ follows from Proposition 2.1.
2. It is easy to find a *subspace* W' of V which is complementary to U. For example, if U has a basis $u_1, \dots u_m$ we extend to a basis $u_1, \dots u_m, u_{m+1}, \dots u_{m+n}$ for V and then let W' equal the span of $u_{m+1}, \dots u_{m+n}$. The content of the Theorem is that we can find a G-stable subspace (a.k.a subrepresentation) W of V which is complementary to U.
3. The Theorem fails for infinite groups. For example, consider $G = \mathbb{Z}$, the representation of $\mathbb{Z}$ on $\mathbb{C}^2$ given by
$$1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
and the one-dimensional subrepresentation $U = \mathbb{C}.(1, 0)$. There is no $\mathbb{Z}$-stable subspace of $\mathbb{C}^2$ which is complementary to U.

Before we start proving the Theorem, let's give two corollaries.

Corollary 2.1. *An indecomposable representation of a finite group G is irreducible.*

Proof. Suppose, for a contradiction, that V is indecomposable but reducible. So we have a subrepresentation $U \subset V$ which is not equal to $\{0\}$ or V. Maschke's theorem tells us that we have a complementary subrepresentation $W \subset V$ and $V \cong U \oplus W$. Since U is not equal to V, W is non-zero, and we have shown that V is actually decomposable. This is a contradiction and so we have proved that indecomposable implies irreducible. □

Corollary 2.2. *Every (finite-dimensional) representation V of a finite group G is isomorphic to a direct sum*

$$V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

with each V_i an irreducible representation of G.

Proof. We induct on the dimension of V. It is obvious that a one-dimensional representation is irreducible. Now let V have dimension n and suppose that every representation of dimension $< n$ is isomorphic to a direct sum of irreducible representations. If V is irreducible we are done. Otherwise, we let $\{0\} \neq U \subsetneq V$ be a proper subrepresentation. Maschke's theorem implies that $V \cong U \oplus W$ for some subrepresentation W of V, and both U and W have dimension strictly less than n. By the inductive hypothesis, U and W are isomorphic to direct sums of irreducible representations. Therefore V is also isomorphic to a direct sum of irreducible representations. □

Lecture 6

In order to prove Maschke's theorem, we have to find a way to come up with complementary subspaces. The next Lemma gives us a way to do that:

Lemma 2.1. *Let V be a vector space and suppose we have a linear map*

$$f : V \to V$$

such that $f \circ f = f$. Then $\ker(f) \subset V$ is a complementary subspace to $\mathrm{im}(f) \subset V$, so $V \cong \ker(f) \oplus \mathrm{im}(f)$.

If V is a representation of a group G and $f : V \to V$ is a G-linear map, then V is isomorphic to $\ker(f) \oplus \mathrm{im}(f)$ as representations of G.

Proof. The rank–nullity theorem says that $\dim(\ker f) + \dim(\mathrm{im}\, f) = \dim(V)$. Suppose $v \in \mathrm{im}(f) \cap \ker(f)$. Since $v \in \mathrm{im}(f)$ we can write $v = f(u)$ and so $f(v) = f \circ f(u) = f(u) = v$. But we also have $v \in \ker(f)$, so $f(v) = 0$. We conclude that $v = 0$. So $\mathrm{im}(f) \cap \ker(f) = \{0\}$. We conclude that $\ker f$ and $\mathrm{im}\, f$ are complementary subspaces.

Alternatively, consider the linear map $v \mapsto (v - f(v), f(v))$ from V to $\ker(f) \oplus \mathrm{im}(f)$. It is an injective map, since if $f(v) = 0$ then $v - f(v) = v$ so if both $f(v)$ and $v - f(v)$ are zero then $v = 0$. It is also surjective since the map $(u, w) \mapsto u + w$ gives an inverse.

If V is a representation of G then, since $\ker f$ and $\mathrm{im}\, f$ are subrepresentations of V which are complementary subspaces, we get the desired isomorphism of G-representations. □

Definition 2.2. Let $f : V \to V$ be a linear map with $f \circ f = f$. Then we say that f is a projection. If V is a representation of G and f is G-linear, we say that f is a G-linear projection.

Example 2.1. Let $V = U \oplus W$ and let $\pi_U : V \to V$ be the map given by $(u, w) \mapsto u$. Then π_U is a projection, with $\ker \pi_U = W$ and $\mathrm{im}\, \pi_U = U$. The above Lemma says that all projections look like this, after composing with an isomorphism.

Example 2.2. Here's a general example of a G-linear projection. Let G be a finite group (finiteness is crucial here) and let V be any representation of G.

We define a map $e_1 : V \to V$ by

$$e_1(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g) v.$$

Then e_1 is a G-linear projection with image the invariants

$$V^G = \{v \in V | \rho_V(g)v = v \forall g \in G\}.$$

To check this we first check that e_1 is G-linear and that e_1 has image contained in V^G. Checking e_1 is G-linear is similar to checking that the image is contained in V^G so we just do the second check. Suppose that $h \in G$ and $v \in V$. Then

$$\rho_V(h) e_1(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(hg) v.$$

The right hand side can be rewritten as

$$\frac{1}{|G|} \sum_{g \in G} \rho_V(g) v = e_1(v)$$

and so $e_1(v) \in V^G$.

Next we check that the restriction $e_1|_{V^G}$ is the identity. This shows that $\operatorname{im} e_1 \supset V^G$. Combined with the previous observation that $\operatorname{im} e_1 \subset V^G$ we get that $\operatorname{im} e_1 = V^G$. Then we also get that $e_1 \circ e_1 = e_1$ since $e_1(e_1(v)) = e_1(v)$.

Definition 2.3. Let V, W be two representations of G and let $\operatorname{Hom}_{\mathbb{C}}(V, W)$ be the vector space of linear maps (*not* just the G-linear maps) between them. We make this into a representation of G by defining

$$(g \cdot f)v = \rho_W(g) f(\rho_V(g^{-1})v).$$

Lemma 2.2. *A linear map $f \in \operatorname{Hom}_{\mathbb{C}}(V, W)$ is G-linear if and only if $g \cdot f = f$ for all $g \in G$. In other words,* $\operatorname{Hom}_{\mathbb{C}}(V, W) = \operatorname{Hom}_G(V, W)$.

Proof. An exercise. $\square$

Corollary 2.3. *V, W as before. Let $f \in \operatorname{Hom}_{\mathbb{C}}(V, W)$. Then $e_1(f) \in \operatorname{Hom}_G(V, W)$. Note that*

$$e_1(f) : v \mapsto \frac{1}{|G|} \sum_{g \in G} \rho_W(g) f(\rho_V(g^{-1})v).$$

Proof. This combines Example 2.2 with Lemma 2.2. $\square$

Proof of Maschke's Theorem, Theorem 2.1. Recall what we want to prove: Let V be a representation of a finite group G. Let $U \subset V$ be a subrepresentation. Then there exists a complementary subrepresentation $W \subset V$ to U.

Proof. First we pick a complementary subspace (not necessarily a subrepresentation!) W' to U. For example, we can pick a basis $v_1, \dots v_m$ for U, extend to a basis $v_1, \dots v_m, v_{m+1}, \dots, v_{m+n}$ for V and then let W' be the subspace spanned by $v_{m+1}, \dots, v_{m+n}$. This gives us a projection map $\pi_U : V \to V$ with image U: if $v = \sum_{i=1}^{m+n} \lambda_i v_i$ we define $\pi_U(v) = \sum_{i=1}^{m} \lambda_i v_i$. However, π_U is not necessarily G-linear. But Corollary 2.3 tells us how to produce a G-linear map: we set $\pi = e_1 \pi_U : V \to V$.

We claim that π is a G-linear projection with image U. If this claim is correct, then we have proved the theorem, since we can take $W = \ker(\pi)$. First we check that π has image contained in U. If $v \in V$ then

$$e_1 \pi_U(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \left(\pi_U(\rho_V(g^{-1})v) \right).$$

Since π_U has image equal to U, and U is stable under the action of G, this is a linear combination of vectors in U, hence is in U.

Now we check that the restriction of π to U is the identity. Suppose $u \in U$. Then

$$\pi(u) = e_1 \pi_U(u) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \left(\pi_U(\rho_V(g^{-1})u) \right).$$

Since $u \in U$, and therefore $\rho_V(g^{-1})u \in U$, we have $\pi_U(\rho_V(g^{-1})u) = \rho_V(g^{-1})u$, so

$$\pi(u) = \frac{1}{|G|} \sum_{g \in G} u = u.$$

This shows that $\pi \circ \pi(v) = \pi(v)$.

As in Example 2.2 we conclude that π is a G-linear projection with image U. $\square$

2.2. Schur's lemma and Abelian groups.

Theorem 2.2 (Schur's lemma)**.** *Let V and W be irreducible reps of G.*

1. *Let $f : V \to W$ be a G-linear map. Then f is either an isomorphism or the zero map.*
2. *Let $f : V \to V$ be a G-linear map. Then $f = \lambda \, \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.*
3. *If V and W are isomorphic then*

$$\dim \mathrm{Hom}_G(V, W) = 1$$

otherwise, $\mathrm{Hom}_G(V, W) = \{0\}$.

Proof. (1) Suppose f is not the zero map. We are going to show that f is an isomorphism. The image of f is a subrepresentation of W, and it is non-zero since f is non-zero. Since W is irreducible, the image of f must be all of W. So f is surjective.

Similarly, the kernel of f is a subrepresentation of V. Since f is non-zero, the kernel of f is not all of V. Since V is irreducible, the kernel of f must be $\{0\}$. So f is injective. We have shown that f is a bijective G-linear map, so it is an isomorphism of representations.

(2) Since V is a finite dimensional complex vector space, every linear map from V to V has at least one eigenvalue. Let λ be an eigenvalue of f. Consider the map

$$f - \lambda \operatorname{id}_V : V \to V.$$

Since f and $\lambda \operatorname{id}_V$ are G-linear, their difference $f - \lambda \operatorname{id}_V$ is G-linear. Now part 1 implies that $f - \lambda \operatorname{id}_V$ is either the zero map or an isomorphism. Since λ is an eigenvalue of f the kernel of $f - \lambda \operatorname{id}_V$ is non-zero. Therefore $f - \lambda \operatorname{id}_V$ is not an isomorphism, and must be the zero map. So $f = \lambda \operatorname{id}_V$.

(3) This follows from parts (1) and (2).

□

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Schur's lemma allows us to give a proof of Fact 1.1.

Corollary 2.4. *Suppose G is a finite Abelian group. Then every irreducible representation of G is one-dimensional.*

Proof. Let V, ρ_V be an irreducible representation of G. Pick any h in G and consider

$$\rho_V(h) : V \to V.$$

Since G is Abelian, the map $\rho_V(h)$ is G-linear. Indeed, for $g \in G$ we have

$$\rho_V(h) \circ \rho_V(g) = \rho_V(hg) = \rho_V(gh) = \rho_V(g) \circ \rho_V(h),$$

so $\rho_V(h)$ is G-linear.

By Schur's lemma, $\rho(h) = \lambda_h \operatorname{id}_V$ for some $\lambda_h \in \mathbb{C}$. So every element of G acts on V as multiplication by a scalar in $\mathbb{C}$. In particular, every vector subspace of V is a subrepresentation of V. Since V is also irreducible, it must be one-dimensional. □

We can now described all the irreps of Abelian groups (and using Theorem 2.1 this gives a description of all the representations of Abelian groups). First we start with cyclic groups:

Exercise 2.1. Let V be an irreducible representation of $C_n = \{e, g, \ldots, g^{n-1}\}$. Then there exists $\zeta \in \mathbb{C}$ with $\zeta^n = 1$ such that V is isomorphic to the one-dimensional representation given by $\mathbb{C}$ with g acting as multiplication by ζ.

The n irreducible representations given by choosing

$$\zeta = 1, \zeta = e^{\frac{2\pi i}{n}} \ldots \zeta = e^{\frac{2\pi i}{n}(n-1)}$$

are all non-isomorphic, so there are n isomorphism classes of irreducible representations of C_n.

Fact 2.1. Every finite Abelian group is isomorphic to a product of cyclic groups.

So now we suppose that $G = C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r}$. Let's denote a generator of C_{n_j} by g_j. For integers $0 \le k_1 \le n_1 - 1$, $0 \le k_2 \le n_2 - 1, \ldots$, $0 \le k_r \le k_r - 1$ we can define an irreducible representation of G by letting g_j act as multiplication by $e^{\frac{2\pi i}{n_j}(k_j)}$.

This gives $|G| = n_1 n_2 \cdots n_r$ non-isomorphic irreducible representations of G, and every irreducible representation of G is isomorphic to one of these representations.

2.3. **Uniqueness of decomposition into irreps.** We've shown that every rep V of a finite group G is isomorphic to a direct sum of irreps. In this subsection we're going to prove that this decomposition as a direct sum of irreps is unique, up to reordering and isomorphism.

Lemma 2.3. *Suppose $V = V_1 \oplus V_2 \oplus \cdots V_d = \bigoplus_{i=1}^d V_i$ with each V_i a rep of G. Let W be any rep of G. Then*

(1) $\mathrm{Hom}_{\mathbb{C}}(W, V) = \bigoplus_{i=1}^d \mathrm{Hom}_{\mathbb{C}}(W, V_i)$ *(more precisely there is a natural isomorphism between these two complex vector spaces)*
(2) $\mathrm{Hom}_{\mathbb{C}}(V, W) = \bigoplus_{i=1}^d \mathrm{Hom}_{\mathbb{C}}(V_i, W)$
(3) $\mathrm{Hom}_G(W, V) = \bigoplus_{i=1}^d \mathrm{Hom}_G(W, V_i)$
(4) $\mathrm{Hom}_G(V, W) = \bigoplus_{i=1}^d \mathrm{Hom}_G(V_i, W)$

Proof. Let's write down the natural isomorphism between the two sides of part (1). Suppose f is a linear map from W to V. Then for each $i = 1, \dots, d$ we get a linear map $f_i : W \to V_i$ by taking f and composing with the map $\pi_i : V \to V_i$ which picks out the ith component. So we define a map

$$\alpha : \mathrm{Hom}_{\mathbb{C}}(W, V) \to \bigoplus_{i=1}^d \mathrm{Hom}_{\mathbb{C}}(W, V_i)$$

by setting

$$\alpha(f) = (f_1, f_2, \dots, f_d).$$

Now we just need to check this map is invertible. Indeed we can write down the inverse

$$\alpha^{-1} : \bigoplus_{i=1}^d \mathrm{Hom}_{\mathbb{C}}(W, V_i) \to \mathrm{Hom}_{\mathbb{C}}(W, V)$$

by defining $\alpha^{-1}(f_1, f_2, \dots, f_d)$ to be the linear map from W to V which takes w to $(f_1(w), f_2(w), \dots, f_d(w))$.

Part (2) is proved in a similar way. Suppose λ is a linear map from V to W. Then for each $i = 1, \dots, d$ we get a linear map $\lambda_i : V_i \to W$ by composing λ with the map $\iota_i : V_i \to V$ which takes v to v in the ith component and 0 in all the other components.

Then the natural isomorphism

$$\beta : \mathrm{Hom}_{\mathbb{C}}(V, W) \to \bigoplus_{i=1}^d \mathrm{Hom}_{\mathbb{C}}(V_i, W)$$

is given by setting

$$\beta(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_d).$$

To show part (3), we just need to show that α and α^{-1} both take G-linear maps to G-linear maps, which is a straightforward check. Likewise, part (4) follows from part (2). Alternatively we can deduce parts (3) and (4) from the statements of parts (1) and (2), by applying the projection map e_1 from Example 2.2. ☐

Proposition 2.2. *Let V be a rep of a group G, with $V \cong V_1 \oplus V_2 \oplus \cdots \oplus V_r$ and each V_i an irrep of G. Let W be an irrep of G. Then the number of V_i which are isomorphic to W is equal to $\dim_{\mathbb{C}} \operatorname{Hom}_G(W, V)$ which also equals $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W)$.*

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Proof. By Lemma 2.3 we have an isomorphism

$$\operatorname{Hom}_G(W, V) \cong \bigoplus_{i=1}^{r} \operatorname{Hom}_G(W, V_i)$$

so equating the dimensions of both sides we get

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(W, V) = \sum_{i=1}^{r} \dim_{\mathbb{C}} \operatorname{Hom}_G(W, V_i).$$

Theorem 2.2 (Schur's lemma) now tells us that the right hand side is the number of V_i which are isomorphic to W.

A very similar argument, beginning with the isomorphism

$$\operatorname{Hom}_G(V, W) \cong \bigoplus_{i=1}^{r} \operatorname{Hom}_G(V_i, W)$$

shows that this number also equals $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W)$. □

Corollary 2.5. *Let V be a rep of G and suppose that we have*

$$V \cong V_1 \oplus \cdots \oplus V_r$$

and

$$V \cong W_1 \oplus \cdots \oplus W_s$$

with the V_i and W_j all irreps. Then $r = s$ and the decompositions are the same up to reordering and isomorphism. In other words, there is a permutation σ of $\{1, \ldots, d\}$ such that $V_i \cong W_{\sigma i}$ for $i = 1, \ldots, d$.

Proof. Let W be any irrep of G. By Proposition 2.2, the number of V_i which are isomorphic to W is equal to $\dim_{\mathbb{C}} \operatorname{Hom}_G(W, V)$. But this number is also the number of W_j which are isomorphic to W. This proves the Corollary. □

Remark 2.2. We have proved that for any rep V of a finite group G, we have

$$V \cong \bigoplus_{W} W^{\oplus \dim_{\mathbb{C}} \operatorname{Hom}_G(V,W)}$$

where the direct sum runs over distinct representatives for the isomorphism classes of irreps of G. The notation is that $W^{\oplus \dim_{\mathbb{C}} \operatorname{Hom}_G(V,W)}$ is a direct sum of $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W)$ copies of W. If $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = 0$ then the notation $W^{\oplus 0}$ means that we take the zero vector space (or ignore the rep W, since it does not show up in the decomposition of V into irreducibles).

2.4. **Decomposition of the regular representation.** We are going to investigate the decomposition of the regular representation $\mathbb{C}G$ of a finite group G. Since

$$\mathbb{C}G \cong \bigoplus_W W^{\oplus \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathbb{C}G, W)}$$

we need to investigate the dimension of the vector spaces $\mathrm{Hom}_G(\mathbb{C}G, W)$ for irreps W.

Proposition 2.3. *Let V be a representation of G. The map*

$$ev_e : \mathrm{Hom}_G(\mathbb{C}G, V) \to V$$

given by taking a G-linear map f to $f([e])$ is an isomorphism of complex vector spaces.

Proof. We check the map is injective and surjective. First injectivity: suppose that $f([e]) = 0$. This implies that $\rho_V(g)f([e]) = 0$ for all $g \in G$. Since f is G-linear, we conclude that $f(\rho_{\mathbb{C}G}[e]) = f([g]) = 0$ for all $g \in G$. So $f = 0$ (since it is zero on every element of a basis for $\mathbb{C}G$).

Now surjectivity: let $v \in V$. We define $f_v : \mathbb{C}G \to V$ by $f_v([g]) = \rho_V(g)v$. This f_v is a G-linear map, since

$$f_v(\rho_{\mathbb{C}G}(h)[g]) = f_v([hg]) = \rho_V(hg)v = \rho_V(h)f_v([g]).$$

Moreover, we have $ev_e(f_v) = \rho_V(e)v = v$, so f_v is a preimage of v. □

Corollary 2.6. *Let G be a finite group.*

(1) *We have a G-linear isomorphism*

$$\mathbb{C}G \cong \bigoplus_W W^{\dim_{\mathbb{C}} W}$$

where the direct sum runs over distinct representatives for the isomorphism classes of irreps of G.

(2) *There are finitely many isomorphism classes of irreps of G.*

(3) *Let $W_1, \ldots, W_r$ be distinct representatives for the isomorphism classes of irreps of G. Set $d_i = \dim_{\mathbb{C}} W_i$. Then*

$$|G| = \sum_{i=1}^{r} d_i^2.$$

Proof. The above proposition shows that for any irrep W, $\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathbb{C}G, W) = \dim_{\mathbb{C}} W$. Applying Remark 2.2 gives part 1.

We deduce part 2 from part 1, since $\mathbb{C}G$ is finite dimensional.

Finally, part 3 also follows from part 1: it says the dimensions of the isomorphic representations $\mathbb{C}G$ and $\bigoplus_W W^{\dim_{\mathbb{C}} W}$ are equal. □

Example 2.3. (1) First we let G be any finite Abelian group. We have shown that the number of isomorphism classes of irreps of G is equal to $|G|$, and they are all one-dimensional. So the equality in Corollary 2.6 reads

$$|G| = \sum_{i=1}^{|G|} 1^2.$$

(2) Now we consider $G = S_3$. On Problem Sheet 2, we say that S_n has two distinct isomorphism classes of one-dimensional representation. So for $G = S_3$ the equality in Corollary 2.6 reads

$$6 = 1^2 + 1^2 + d_3^2 + \cdots + d_r^2$$

with $d_3, \ldots, d_r$ all ≥ 2. We can conclude from this that in fact $r = 3$ and the dimensions of the three isomorphism classes of irreps are 1, 1 and 2. We already found an irreducible two-dimensional representation (Example 1.7), so now we have proved that every irrep of S_3 is isomorphic to one of: the trivial rep of dimension one, the one-dimensional rep given by the sign homomorphism, the two-dimensional rep defined in Example 1.7.

2.5. **Duals and tensor products.** Warning: I have reordered the material here compared to how I lectured it. I think what's written here should be clearer for those of you who aren't familiar with dual vector spaces.

Earlier in the course we considered the complex vector space of linear maps $\mathrm{Hom}_{\mathbb{C}}(V, W)$, where V and W are two complex vector spaces. If V and W are representations of a group G, we defined a representation of G on the vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ (see Definition 2.3).

We're going to consider a special case of this construction, where W is the vector space $\mathbb{C}$ with the trivial action of G (i.e. W is the one-dimensional trivial rep of G).

Definition 2.4. If V is a complex vector space, let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the *dual* vector space, of linear maps from V to $\mathbb{C}$.

If V is a representation of G, we define a representation of G on the vector space V^* by setting

$$\rho_{V^*}(g)(f) = f \circ \rho_V(g^{-1})$$

for $f \in V^*$ and $g \in G$. In other words, $\rho_{V^*}(g)(f)$ is the linear map from V to $\mathbb{C}$ given by $v \mapsto f(\rho_V(g^{-1})v)$.

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Example 2.4. Suppose V is the n-dimensional vector space of column vectors of length n. Then V^* can be identified with the n-dimensional vector space of row vectors of length n. A row vector $\mathbf{x}$ is viewed as a linear map from V to $\mathbb{C}$ by taking v to the matrix product $\mathbf{x}v$ (i.e. the scalar product).

Example 2.5. Let $B = b_1, \ldots, b_n$ be a basis of V. For each $i = 1, \ldots, n$ we define an element $\delta_i \in V^*$ by $\delta_i(b_j) = 0$ if $i \neq j$ and $\delta_i(b_i) = 1$. Let B^* denote the elements $\delta_1, \ldots, \delta_n$ of V^*. Then this is a basis of V^*, called the *dual basis* to B.

Here's a proof that B^* is a basis. Suppose we have $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and $f = \sum_{i=1}^n \lambda_i \delta_i = 0$. Applying f to b_j we get $f(b_j) = \lambda_j = 0$. So $\lambda_i = 0$ for all i, and we have shown that B^* is a linearly independent set of vectors in V^*.

Now we show that B^* spans V^*. Suppose $f \in V^*$. Set $\lambda_i = f(b_i)$. Then consider the linear combination $f' = \sum_{i=1}^n \lambda_i \delta_i$. We have $f'(b_j) = \lambda_j = f(b_j)$ for all j, and so $f' = f$. This shows that f is in the span of B^*. So we have shown that B^* is a basis for V^*.

Example 2.6. Let's work out the dual of a one-dimensional representation. Suppose V is a one-dimensional vector space. Then we have a homomorphism

$$\chi : G \to \mathbb{C}^\times$$

such that $\rho_V(g)v = \chi(g)v$ for all $v \in V$. We'll write V_χ for this representation, to remember the action of G.

Now consider the dual space V_χ^* and let $\delta \in V_\chi^*$. We have $\rho_{V_\chi^*}(g)\delta = \delta \circ \rho_{V_\chi}(g^{-1})$. Since $\rho_{V_\chi}(g^{-1})$ is multiplication by $\chi(g)^{-1}$ we have $\rho_{V_\chi{}^*}(g)\delta = \chi(g)^{-1}\delta$, so the one-dimensional representation V_χ^* has action of G given by the homomorphism χ^{-1}. We can also write $\overline{\chi}$ for this character, since for each g $\chi(g)$ is a root of unity: this implies that $\chi(g^{-1})$ is equal to $\overline{\chi}(g)$, the complex conjugate of $\chi(g)$.

Exercise 2.2. Let V be a finite dimensional vector space. Consider the map

$$\alpha : V \to (V^*)^*$$

defined by letting $\alpha(v)$ be the linear map

$$\alpha(v) : V^* \to \mathbb{C}$$

given by $\alpha(v)(\delta) = \delta(v)$, for $\delta \in V^*$. Show that this map is an isomorphism of vector spaces.

If V is a representation of G, show that α is a G-linear isomorphism.

Proposition 2.4. *Let V be a (finite-dimensional) representation of a group G. Then V is irreducible if and only if V^* is irreducible.*

Proof. First we show that V reducible implies V^* reducible. Suppose $V \cong U \oplus W$. Then (the proof of) Lemma 2.3 gives us an isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C}) \oplus \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$$

or in other words

$$V^* \cong U^* \oplus W^*$$

and it's easy to check this isomorphism is G-linear. We deduce that V^* is reducible.

Now suppose V^* is reducible. We have just shown that $(V^*)^*$ is reducible. Exercise 2.2 tells us that V is reducible (since $V \cong (V^*)^*$).

Therefore we have shown that V is reducible if and only if V^* is reducible. Equivalently, V is irreducible if and only if V^* is irreducible. □

Suppose V is a rep of G and B is a basis of V. We are going to compute the matrix rep given by V^* with respect to the dual basis B^*. First we are going to do a little linear algebra:

Suppose V is a vector space, and $f : V \to V$ is a linear map. Then we define the dual map $f^* : V^* \to V^*$ by $f^*(\delta) = \delta \circ f$. In other words we define $f^*(\delta)$ to be the linear map from V to $\mathbb{C}$ which takes v to $\delta(f(v))$. Fix a basis $B = b_1, \ldots, b_n$ of V and denote the matrix $[f]_B$ by M. Write $B^* = \delta_1, \ldots, \delta_n$ for the dual basis of V^*.

Claim. *The matrix $[f^*]_{B^*}$ is the transpose M^t of the matrix M: recall that $(M^t)_{i,j} = M_{j,i}$.*

Proof. Let's consider $f^*(\delta_i)$. We have

$$f^*(\delta_i)(b_j) = \delta_i(f(b_j)) = \delta_i(\sum_{k=1}^{n} M_{k,j} b_k) = M_{i,j}.$$

We conclude that

$$f^*(\delta_i) = \sum_{j=1}^{n} M_{i,j}\delta_j$$

which shows that $[f^*]_{B^*} = M^t$. □

We can now show the following Proposition:

Proposition 2.5. *Let V be a rep of G, fix a basis B of V, and write M for the matrix $[\rho_V(g)]_B$. Then*

$$[\rho_{V^*}(g)]_{B*} = (M^{-1})^t.$$

Proof. From the definition of the dual representation, we have $\rho_{V^*}(g) = \rho_V(g^{-1})^*$. Since the matrix $[\rho_V(g^{-1})]_B$ is equal to M^{-1} the Proposition follows from the above Claim. □

Remark 2.3. We write M^{-t} for the matrix $(M^{-1})^t$. In fact, we also have $M^{-t} = (M^t)^{-1}$, since $M^t(M^{-1})^t = (M^{-1}M)^t$ is equal to the identity matrix, as is $(M^{-1})^t M^t = (MM^{-1})^t$ (we use that $(AB)^t = B^t A^t$).

Example 2.7. Let's give another example of computing the dual representation. Let V be the two-dimensional irrep of S_3 defined in Example 1.7. It follows from Proposition 2.4 that V^* is also a two-dimensional irrep of S_3. But we showed in Example 2.3 that there is only one isomorphism class of two-dimensional irreps of S_3. So V^* is isomorphic to V!

We can give a different proof that V^* is isomorphic to V using the computation of the matrix rep. Recall that we defined V (or rather, the associated matrix rep with respect to the standard basis of $\mathbb{C}^2$) by $\rho_V(123) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$, $\rho_V(23) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. To give the matrix rep for V^* we just need to compute the matrices $\rho_V(123)^{-t}$ and $\rho_V(23)^{-t}$. But if you do the computation, you see that these matrices are equal to their own inverse transpose. So we get exactly the same matrix rep for V^*, which shows that V and V^* are isomorphic.

Now we're moving on to tensor products.

Definition 2.5. Let V and W be two vector spaces, with a basis $A = a_1, \ldots, a_m$ for V and a basis $B = b_1, \ldots, b_n$ for W. We define $V \otimes W$ to be the mn-dimensional vector space with basis (denoted $A \otimes B$) given by the symbols $a_i \otimes b_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

If $v = \sum_{i=1}^m \lambda_i a_i \in V$ and $w = \sum_{j=1}^n \mu_j b_j \in W$ we define $v \otimes w$ to be the element of $V \otimes W$ given by

$$v \otimes w = \sum_{i,j} \lambda_i \mu_j (a_i \otimes b_j).$$

Warning. Not every element of $V \otimes W$ is of the form $v \otimes w$ for $v \in V$ and $w \in W$. For example, suppose $m = n = 2$ and consider $a_1 \otimes b_1 + a_2 \otimes b_2 \in V \otimes W$. This element is not equal to $v \otimes w$ for any $v \in V$, $w \in W$.

Lecture 10

Definition 2.6. Let V and W be representations of G with bases A, B as before. We define linear maps

$$\rho_{V\otimes W}(g) : V \otimes W \to V \otimes W$$

by

$$\rho_{V\otimes W}(g) : a_i \otimes b_j \mapsto \rho_V(g)a_i \otimes \rho_V(g)b_j.$$

We will eventually show that this defines a representation of G on the vector space $V \otimes W$. First we'll describe the matrix corresponding to the linear map $\rho_{V\otimes W}(g)$ with respect to the basis $A \otimes B$.

Let $M = [\rho_V(g)]_A$ and $N = [\rho_W(g)]_B$. We're going to write $M \otimes N$ for the matrix corresponding to the linear map $\rho_{V\otimes W}(g)$ with respect to the basis $A \otimes B$.

It's simplest to refer to specify entries of the matrix $M \otimes N$ by a pair $(i, j), (s, t)$, where $1 \leq i, s \leq m$ and $1 \leq j, t \leq n$. This means that we have matrix entries $(M \otimes N)_{(i,j),(s,t)}$ which satisfy

$$\rho_{V\otimes W}(g)(a_s \otimes b_t) = \sum_{i,j} (M \otimes N)_{(i,j),(s,t)} a_i \otimes b_j.$$

Lemma 2.4. *With the above notation, we have*

$$(M \otimes N)_{(i,j),(s,t)} = M_{is} N_{jt}.$$

Proof. By definition we have

$$\begin{aligned}
\rho_{V\otimes W}(g)(a_s \otimes b_t) &= \rho_V(g)a_s \otimes \rho_W(g)b_t \\
&= (\sum_i M_{is} a_i) \otimes (\sum_j N_{jt} b_j) \\
&= \sum_{i,j} M_{is} N_{jt} (a_i \otimes b_j).
\end{aligned}$$

This proves the lemma. □

Proposition 2.6. *Let V and W be reps of G, with bases A and B. Then there exists an isomorphism of vector spaces*

$$\alpha : V \otimes W \cong \mathrm{Hom}_{\mathbb{C}}(V^*, W)$$

with

$$\alpha \circ \rho_{V\otimes W}(g) = \rho_{\mathrm{Hom}_{\mathbb{C}}(V^*,W)}(g) \circ \alpha$$

for all $g \in G$.

Proof. We're going to start by writing down a basis for $\mathrm{Hom}_{\mathbb{C}}(V^*, W)$. We have a dual basis $A^* = \delta_1, \ldots, \delta_m$ for V^*. For $1 \leq i \leq m$ and $1 \leq j \leq n$ we define $f_{ij} \in \mathrm{Hom}_{\mathbb{C}}(V^*, W)$ by $f_{ij}(\delta_k) = 0$ if $k \neq i$ and $f_{ij}(\delta_i) = b_j$.

Let's show that this gives a basis for $\mathrm{Hom}_{\mathbb{C}}(V^*, W)$. If we consider the matrix $[f]_{A^*,B}$ associated to $f \in \mathrm{Hom}_{\mathbb{C}}(V^*, W)$ we get an isomorphism of vector spaces

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}}(V^*, W) &\to M_{n,m}(\mathbb{C}) \\
f &\mapsto [f]_{A^*,B}.
\end{aligned}$$

A basis for the space of $n \times m$ matrices $M_{n,m}$ is given by taking the matrices with 1 in the (j, i) entry and 0 everywhere else (for i,j varying with $1 \leq i \leq m$ and $1 \leq j \leq n$). Under our isomorphism these basis elements correspond to the f_{ij}. So the f_{ij} form a basis.

Now we define the map α by $\alpha(a_i \otimes b_j) = f_{ij}$. This is an isomorphism of vetor spaces, since it takes a basis to a basis. We now need to check the claim that

$$\alpha \circ \rho_{V\otimes W}(g) = \rho_{\mathrm{Hom}_{\mathbb{C}}(V^*,W)}(g) \circ \alpha$$

for all $g \in G$.

First we compute $\alpha \circ \rho_{V\otimes W}(g)(a_s \otimes b_t)$. As usual we write $M = [\rho_V(g)]_A$ and $N = [\rho_W(g)]_B$. So we have

$$\rho_{V\otimes W}(g)(a_s \otimes b_t) = \rho_V(g)a_s \otimes \rho_W(g)b_t = \sum_{i,j} M_{is}N_{jt}a_i \otimes b_j$$

by Lemma 2.4. So we have

$$\alpha \circ \rho_{V\otimes W}(g)(a_s \otimes b_t) = \sum_{i,j} M_{is}N_{jt}f_{ij}.$$

Now we need to show that

$$\rho_{\mathrm{Hom}_{\mathbb{C}}(V^*,W)}(g)f_{st} = \sum_{i,j} M_{is}N_{jt}f_{ij}.$$

Equivalently, we need to show that for $1 \leq i \leq m$ we have

$$(\rho_{\mathrm{Hom}_{\mathbb{C}}(V^*,W)}(g)f_{st})(\delta_i) = \sum_{j} M_{is}N_{jt}b_j.$$

So now we compute $(\rho_{\mathrm{Hom}_{\mathbb{C}}(V^*,W)}(g)f_{st})(\delta_i)$. By definition we have

$$\rho_{\mathrm{Hom}_{\mathbb{C}}(V^*,W)}(g)f_{st} = \rho_W(g) \circ f_{st} \circ \rho_{V^*}(g^{-1}).$$

We have

$$\rho_{V^*}(g^{-1})\delta_i = \sum_{k=1}^{m} M_{ik}\delta_k$$

by Proposition 2.5. Now we apply f_{st}, to get

$$f_{st} \circ \rho_{V^*}(g^{-1})\delta_i = M_{is}b_t.$$

Finally, we apply $\rho_W(g)$ to get

$$(\rho_{\mathrm{Hom}_{\mathbb{C}}(V^*,W)}(g)f_{st})(\delta_i) = M_{is}\rho_W(g)b_t = \sum_{j} M_{is}N_{jt}b_j$$

as desired. □

Corollary 2.7. *Let V, W be reps of G and fix bases A, B of V, W respectively.*

1. *The map $g \mapsto \rho_{V\otimes W}(g)$ defines a representation of G on $V \otimes W$.*
2. *The map α in the above Proposition gives a G-linear isomorphism between $V \otimes W$ and $\mathrm{Hom}_{\mathbb{C}}(V^*, W)$.*

(3) *If we change the bases A, B, the new representation $V \otimes W$ (recall that the definition of this representation used our fixed bases) is isomorphic to one defined by the bases A, B.*

Proof. The first two parts follow immediately from the Proposition. The third part follows from the second part: whatever choice of bases we make, we get a representation which is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(V^*, W)$. □

Lecture 11

Corollary 2.8. *Let V and W be representations of G. Then $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is isomorphic (as a representation of G) to $V^* \otimes W$.*

Proof. By Corollary 2.7, we have an isomorphism

$$V^* \otimes W \cong \mathrm{Hom}_{\mathbb{C}}((V^*)^*, W).$$

By Exercise 2.2, we also have an isomorphism (of reps of G) $(V^*)^* \cong V$, so we have an isomorphism

$$\mathrm{Hom}_{\mathbb{C}}((V^*)^*, W) \cong \mathrm{Hom}_{\mathbb{C}}(V, W).$$

So coming Corollary 2.7 and Exercise 2.2 we get the desired isomorphism

$$V^* \otimes W \cong \mathrm{Hom}_{\mathbb{C}}(V, W).$$

□

Now let's think about some examples of the tensor product construction. We'll suppose that $\dim W = 1$, with a basis vector b_1. A representation of G on W is given by a homomorphism $\chi : G \to \mathbb{C}^\times$: we have $\rho_W(g)w = \chi(g)w$ for $w \in W$ and $g \in G$.

By definition, $V \otimes W$ has a basis $a_1 \otimes b_1, \dots, a_m \otimes b_1$, and the action of G is given by

$$\rho_{V\otimes W}(g)a_i \otimes b_1 = \chi(g)\rho_V(g)a_i \otimes b_1.$$

So the matrix representation with respect to the basis $A \otimes B$ is given by

$$[\rho_{V\otimes W}(g)]_{A\otimes B} = \chi(g)[\rho_V(g)]_A.$$

Example 2.8. (1) To give a more concrete example, let's suppose $\dim V$ is also equal to 1. So we have two homomorphisms χ_V, χ_W which are the one-dimensional matrix representations associated to V and W. Then $V \otimes W$ is a one-dimensional representation whose associated matrix representation is the homomorphism

$$\begin{aligned} G &\to \mathbb{C}^\times \\ g &\mapsto \chi_1(g)\chi_2(g) \end{aligned}$$

(2) Let's consider the irreducible two-dimensional rep of S_3, V, given by Example 1.7. Let W be the one-dimensional rep of S_3 given by the sign homomorphism. Then the matrix rep associated to $V \otimes W$ is determined by $(123) \mapsto \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$ and $(23) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Although it looks like this is a different representation to V, in fact it is isomorphic to V (see problem sheet 3).

3. Character theory

3.1. **Traces.** Recall that if M is a $n \times n$ matrix we define the trace $\mathrm{Tr}(M)$ to be the sum of the diagonal entries of M, i.e.

$$\mathrm{Tr}(M) = \sum_{i=1}^{n} M_{ii}$$

One nice property of the trace is that if M, N are two $n \times n$ matrices, then $\mathrm{Tr}(MN) = \mathrm{Tr}(NM)$. As a consequence, if $P \in \mathrm{GL}_n(\mathbb{C})$ is an invertible matrix, then $\mathrm{Tr}(P^{-1}MP) = \mathrm{Tr}(PP^{-1}M) = \mathrm{Tr}(M)$.

Definition 3.1. Let $f : V \to V$ be a linear map. Let B be a basis for V. We define

$$\mathrm{Tr}(f) = \mathrm{Tr}([f]_B)$$

Remark 3.1. Since $\mathrm{Tr}(P^{-1}MP) = \mathrm{Tr}(M)$ the value of $\mathrm{Tr}([f]_B)$ is independent of the choice of basis B for V. So $\mathrm{Tr}(f)$ is well-defined, independently of the choice of B.

Definition 3.2. Let V be a rep of G. The character of V, denoted χ_V is the complex valued function

$$\begin{aligned} \chi_V : G &\to \mathbb{C} \\ g &\mapsto \mathrm{Tr}(\rho_V(g)) \end{aligned}$$

Remark 3.2. In general, χ_V is not a homomorphism, it is just a function from G to $\mathbb{C}$.

Example 3.1.
(1) Assume $\dim_V = 1$. Then the action of G on V is determined by a homomorphism
$$\chi : G \to \mathbb{C}^{\times}.$$
We have $\rho_V(g)v = \chi(g)v$ for all $g \in G$ and $v \in V$. In this case the character χ_V is just equal to χ.
(2) Let's return to example 1.7. We have $\chi_V(123) = -1$ and $\chi_V(23) = 0$. Note that we can't just write down $\chi_V(g)$ for general g from these two calculations, even though (123) and (23) generate S_3, since χ_V is not a homomorphism.

Lecture 12

Lemma 3.1. *Suppose V and W are two isomorphic representations of a group G. Then $\chi_V = \chi_W$.*

Proof. If V is isomorphic to W, then choosing bases A, B for V, W respectively we get equivalent matrix representations (see Corollary 1.1). In other words there is a matrix $P \in \mathrm{GL}_n(\mathbb{C})$ such that

$$P[\rho_V(g)]_A P^{-1} = [\rho_W(g)]_B$$

for all $g \in G$. In particular, $\rho_V(g)$ and $\rho_W(g)$ are conjugate matrices and hence have the same trace, so $\chi_V(g) = \chi_W(g)$. ∎

We will prove the converse, that representations with the same character are isomorphic, later. The next proposition gives some basic properties of characters:

Proposition 3.1. *Let V be a representation of G (a finite group).*

(1) *If $g, h \in G$ are conjugate, i.e. there is a $k \in G$ with $g = khk^{-1}$, then*
$$\chi_V(g) = \chi_V(h).$$
(2) $\chi_V(e) = \dim(V)$
(3) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, *the complex conjugate of $\chi_V(g)$.*
(4) *The absolute value $|\chi_V(g)|$ satisfies*
$$|\chi_V(g)| \leq \dim(V)$$
and we have equality $|\chi_V(g)| = \dim(V)$ if and only if $\rho_V(g) = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbb{C}^\times$.

Proof. (1) Since $g = khk^{-1}$ we have $\rho_V(g) = \rho_V(k)\rho_V(h)\rho_V(k)^{-1}$. The traces of conjugate matrices are equal, so $\chi_V(g) = \chi_V(h)$.

(2) We have $\rho_V(e) = \mathrm{id}_V$ and the trace of id_V is equal to $\dim(V)$.

(3) Since $\rho_V(g)$ is diagonalisable (by Exercise 2 on Problem Sheet 1, or by considering V as a rep of the cyclic group generated by g which therefore decomposes as a direct sum of one dimensional irreps), we have a basis A for V such that the matrix $[\rho_V(g)]_A$ is a diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_d$. So we have $\chi_V(g) = \lambda_1 + \lambda_2 + \cdots + \lambda_d$ and $\chi_V(g^{-1}) = \lambda_1^{-1} + \lambda_2^{-1} + \cdots + \lambda_d^{-1}$. Since $g^n = e$ for some $n \geq 1$, we also have $\lambda_i^n = 1$, so in fact the λ_i are roots of unity and we have $\overline{\lambda_i} = \lambda_i^{-1}$. We conclude that
$$\chi_V(g^{-1}) = \overline{\lambda_1} + \overline{\lambda_2} + \cdots + \overline{\lambda_d} = \overline{\chi_V(g)}.$$

(4) We again use the fact that $\chi_V(g) = \lambda_1 + \lambda_2 + \cdots + \lambda_d$ with each λ_i a root of unity. In particular we have $|\lambda_i| = 1$, so $|\chi_V(g)| \leq \sum_{i=1}^d |\lambda_i| = d$. We have equality if and only if the arguments of all the complex numbers λ_i are equal, which implies that all the λ_i are equal (since they have absolute value 1). This is the case if and only if $\rho_V(g)$ is multiplication by λ for some $\lambda \in \mathbb{C}^\times$.

□

Corollary 3.1. (1) $\rho_V(g) = \mathrm{id}_V$ *if and only if* $\chi_V(g) = \dim(V)$
(2) *V is a faithful rep of G if and only if the set $\{g : \chi_V(g) = \dim(V)\}$ is equal to $\{e\}$.*

Proof. The second part follows from the first part. For the first one, one direction is immediate: if $\rho_V(g) = \mathrm{id}_V$ then the trace of $\rho_V(g)$ is equal to $\dim(V)$.

Conversely, if $\chi_V(g) = \dim(V)$ then part (4) of the Proposition implies that $\rho_V(g) = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbb{C}^\times$. Since the trace of $\lambda\,\mathrm{id}_V$ is $\lambda \dim(V)$ we can conclude that $\lambda = 1$ and $\rho_V(g) = \mathrm{id}_V$. □

Given representations V, W we have defined various other representations. The next proposition computes the characters of these representations in terms of χ_V, χ_W.

Proposition 3.2. *Let V and W be representations of a (finite) group G.*

(1) $\chi_{V \oplus W} = \chi_V + \chi_W$, *where this character is defined by* $(\chi_V + \chi_W)(g) = \chi_V(g) + \chi_W(g)$.
(2) $\chi_{V \otimes W} = \chi_V \chi_W$, *defined by* $(\chi_V\chi_W)(g) = \chi_V(g)\chi_W(g)$.

(3) $\chi_{V^*} = \overline{\chi_V}$, *defined by* $\overline{\chi_V}(g) = \overline{\chi_V(g)}$.
(4) $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)} = \overline{\chi_V}\chi_W$.

Proof. (1) If we fix bases A, B for V, W, then $[\rho_{V\oplus W}(g)]_{A\oplus B}$ is a block diagonal matrix with blocks $[\rho_V(g)]_A$ and $[\rho_W(g)]_B$, so its trace is the sum $\chi_V(g) + \chi_W(g)$.

(2) If we let $M = [\rho_V(g)]_A$ and $N = [\rho_W(g)]_B$ then $[\rho_{V\otimes W}(g)]_{A\otimes B}$ has trace

$$\sum_{i,j} M_{ii}N_{jj}$$

by Lemma 2.4. This is equal to the product

$$(\sum_i M_{ii})(\sum_j N_{jj}) = \mathrm{Tr}(M)\mathrm{Tr}(N) = \chi_V(g)\chi_W(g).$$

(3) Again we set $M = [\rho_V(g)]_A$. Recall that $\rho_{V^*}(g)_A = M^{-t}$, the inverse transpose (Proposition 2.5). Taking the transpose of a matrix doesn't change the trace, so

$$\chi_{V^*}(g) = \mathrm{Tr}(M^{-1}) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$$

where the last equality is by part 3 of Proposition 3.1.

(4) For this part, we use Corollary 2.7: we have a G-linear isomorphism $\mathrm{Hom}_{\mathbb{C}}(V,W) \cong V^* \otimes W$, so applying parts (2) and (3) we get $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)} = \overline{\chi_V}\chi_W$.

□ Lecture 13

Definition 3.3. Suppose V is an irrep of G. We say that the character χ_V is an *irreducible character* of G.

If G is a finite group and $V_1, V_2, \dots, V_r$ is a complete list of non-isomorphic irreps, then the irreducible characters of G are given by $\chi_1 = \chi_{V_1}, \chi_2 = \chi_{V_2} \cdots \chi_{V_r}$.

If V is any representation of G, then we have $\chi_V = \sum_{i=1}^r m_i\chi_i$, where $m_i = \dim \mathrm{Hom}_G(V, V_i)$, by Remark 2.2.

Exercise 3.1. We have $\chi_{\mathbb{C}G} = \sum_{i=1}^r (\dim V_i)\chi_i$. On the other hand, you can compute directly that $\chi_{\mathbb{C}G}(g) = 0$ if $g \neq e$ whilst $\chi_{\mathbb{C}G}(e) = |G|$.

If you think about the case $G = C_n$ then this gives the (perhaps familiar) fact that the sum

$$\sum_{j=1}^r e^{\frac{2\pi i(j-1)}{n}}$$

is equal to 0.

3.2. **Inner product of characters.**

Definition 3.4. Let $C(G)$ be the complex vector space of functions from G to $\mathbb{C}$. We let $C_{cl}(G) \subset C(G)$ be the subspace of functions satisfy $f(kgk^{-1}) = f(g)$ for all $g, k \in G$. In other words $C_{cl}(G)$ consists of functions which are constant on conjugacy classes, which we call *class functions*.

The dimension of $C(G)$ is equal to $|G|$. The dimension of $C_{cl}(G)$ is equal to $\#ccl(G)$, the number of conjugacy classes in G.

Definition 3.5. If $f_1, f_2 \in C(G)$ we define

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

The pairing $\langle , \rangle$ defines a Hermitian inner product on $C(G)$, and on the subspace $C_{cl}(G)$. In other words, it is linear in the first variable, conjugate-linear in the second variable, satisfies $\langle f_2, f_1 \rangle = \overline{\langle f_1, f_2 \rangle}$ and we have

$$\langle f, f \rangle = \frac{1}{|G|} \sum_{g \in G} |f(g)|^2 \geq 0$$

with $\langle f, f \rangle = 0$ if and only if $f = 0$.

Theorem 3.1. *Let V and W be two reps of G. Then*

$$\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(W, V) = \dim \mathrm{Hom}_G(V, W) = \langle \chi_W, \chi_V \rangle.$$

We'll prove this shortly. First we give some corollaries.

Corollary 3.2. *Let V and W be irreps of G. Then $\langle \chi_V, \chi_W \rangle = 1$ if V and W are isomorphic, and $\langle \chi_V, \chi_W \rangle = 0$ if V and W are not isomorphic.*

Proof. This follows from Schur's lemma plus the above Theorem. □

Recall that we are writing $\chi_1, \ldots, \chi_r$ for the characters of the irreps $V_1, \ldots, V_r$.

Corollary 3.3.
(1) *$\chi_1, \ldots, \chi_r$ form an orthonormal subset of $C_{cl}(G)$. In other words, we have $\langle \chi_i, \chi_j \rangle = 0$ if $i \neq j$ and it equals 1 if $i = j$.*
(2) *We have an inequality $r \leq \#ccl(G)$.*
(3) *Let V be any rep of G. We have*

$$V \cong \bigoplus_{i=1}^{r} V_i^{\oplus \langle \chi_V, \chi_i \rangle}$$

and

$$\chi_V = \sum \langle \chi_V, \chi_i \rangle \chi_i.$$

(4) *Let V be a rep of G. V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Proof.
(1) This is immediate from the previous corollary.
(2) Since we have an orthonormal subset of size r in $C_{cl}(G)$, we have

$$r \leq \dim C_{cl}(G) = \#ccl(G).$$

(3) This follows from the Theorem and Remark 2.2.
(4) If V is irreducible then the previous corollary implies that $\langle \chi_V, \chi_V \rangle = 1$. Conversely, if

$$V \cong \sum_i V_i^{\oplus m_i}$$

then $\langle \chi_V, \chi_V \rangle = \sum_i m_i^2$ so $\langle \chi_V, \chi_V \rangle = 1$ implies that we have one $m_i = 1$ and the rest zero. This implies that V is irreducible.

□

Remark 3.3. Note that number (3) of the above corollary implies that if two representations have the same character, then they are isomorphic.

Lecture 14

Now we're going to prove Theorem 3.1.

Proof of Theorem 3.1. Recall from Proposition 3.2 that

$$\chi_V(g)\overline{\chi_W(g)} = \chi_{\mathrm{Hom}_{\mathbb{C}}(W,V)}(g).$$

So we have

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|}\sum_{g\in G}\chi_{\mathrm{Hom}_{\mathbb{C}}(W,V)}(g).$$

We can write $\frac{1}{|G|}\sum_{g\in G}\chi_{\mathrm{Hom}_{\mathbb{C}}(W,V)}(g)$ as the trace of a single linear map, since we have

$$\frac{1}{|G|}\sum_{g\in G}\chi_{\mathrm{Hom}_{\mathbb{C}}(W,V)}(g) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\rho_{\mathrm{Hom}_{\mathbb{C}}(W,V)}(g)) = \mathrm{Tr}\left(\frac{1}{|G|}\sum_{g\in G}\rho_{\mathrm{Hom}_{\mathbb{C}}(W,V)}(g)\right).$$

Now we recall that we have seen the map $\frac{1}{|G|}\sum_{g\in G}\rho_{\mathrm{Hom}_{\mathbb{C}}(W,V)}(g)$ before. It is the G-linear projection from example 2.2

$$e_1 : \mathrm{Hom}_{\mathbb{C}}(W,V) \to \mathrm{Hom}_{\mathbb{C}}(W,V)$$

with image $\mathrm{Hom}_G(W,V)$. To establish the Theorem, we need to prove that $\mathrm{Tr}(e_1) = \dim(\mathrm{Hom}_G(W,V)) = \dim(\mathrm{im}(e_1))$. This follows from Lemma 3.2.

Note that since $\langle \chi_W, \chi_V \rangle = \overline{\langle \chi_V, \chi_W \rangle} = \langle \chi_V, \chi_W \rangle$ we immediately get the other equalities in the statement of Theorem 3.1. □

Lemma 3.2. *Let V be a (finite dimensional) vector space and suppose $f : V \to V$ is a projection. Then* $\mathrm{Tr}(f) = \dim(\mathrm{im}(f))$.

Proof. Recall that since f is a projection we have $V \cong \mathrm{im}(f) \oplus \ker(f)$. In particular, if $v_1, \dots, v_m$ is a basis for $\ker(f)$ and $v_{m+1}, \dots, v_{m+n}$ is a basis for $\mathrm{im}(f)$ then

$$v_1, \dots, v_m, v_{m+1}, \dots, v_{m+n}$$

is a basis for V. Consider the matrix of f with respect to this basis. We have $f(v_i) = 0$ for $i = 0, \dots, m$ and $f(v_i) = v_i$ for $i = m+1, \dots, m+n$. So the trace of this matrix is equal to n, the dimension of $\mathrm{im}(f)$. □

3.3. **Applications of character theory.** We're going to use character theory, particularly Corollary 3.3, to compute some examples of decomposing representations of finite groups into sums of irreducibles.

Example 3.2. First let $G = C_4 = \{e, g, g^2, g^3\}$. Let's consider the two-dimensional representation given by a two-dimensional vector space V with action of G defined by

$$\rho_V(g) = \begin{pmatrix} i & 2 \\ 1 & -i \end{pmatrix}.$$

Note that we also have

$$\rho_V(g^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So we can write down the character χ_V of ρ_V by computing the trace of these matrices. We also know what the irreducible characters of C_4 are. For $j = 1, \ldots, 4$ we have an irreducible one-dimensional representation V_j with $\rho_{V_j}(g)$ equal to multiplication by $e^{2\pi i(j-1)/4} = i^{j-1}$. So the irreducible characters are $\chi_1, \ldots, \chi_4$, where $\chi_j(g^a) = i^{a(j-1)}$. We can collect the information about these characters into a table:

	e	g	g^2	g^3
χ_V	2	0	2	0
χ_j	1	i^{j-1}	$(-1)^{j-1}$	$(-i)^{j-1}$

Now we can work out $\langle \chi_V, \chi_j \rangle$ for each j. We have

$$\langle \chi_V, \chi_j \rangle = \frac{1}{4}(2 + 2(-1)^{j-1})$$

so we conclude that $\langle \chi_V, \chi_j \rangle = 0$ for $j = 2, 4$ and $\langle \chi_V, \chi_j \rangle = 1$ for $j = 1, 3$.

Applying Corollary 3.3 we can deduce that $V \cong V_1 \oplus V_3$. Note that although we haven't explicitly found one dimensional subrepresentations of V which are isomorphic to V_1 or V_3, we have proven that they do exist!

Example 3.3. Now we'll do a slightly more elaborate example. Let $G = S_4$, and consider the usual permutation action of G on $\Omega = \{1, 2, 3, 4\}$. We get a four dimensional representation $\mathbb{C}\Omega$ and we can ask what it's decomposition into irreducibles is. Recall from Example 1.6 that we know that $\mathbb{C}\Omega$ has a one-dimensional subrepresentation W, spanned by the vector $([1] + [2] + [3] + [4]) \in \mathbb{C}\Omega$. Also, W is isomorphic to the one-dimensional trivial representation.

By Maschke's theorem, there is a three dimensional subrepresentation $U \subset \mathbb{C}\Omega$, complementary to W. We're going to show that U is irreducible, so $\mathbb{C}\Omega \cong U \oplus W$ is a decomposition of $\mathbb{C}\Omega$ into irreducible representations of G, one of dimension 1 and the other of dimension 3. We will show U is irreducible by computing it's character χ_U and showing that $\langle \chi_U, \chi_U \rangle = 1$.

Recall that characters are constant on conjugacy classes, so we just need to work out $\chi_U(g)$ for representatives g of each conjugacy class. Again we organise this information in a table. The first row gives an element of each conjugacy class in S_4 (these are given by the cycle type of a permutation). The second row records the size of each conjugacy class, which will be useful for working out the inner product. The other rows give the values of the characters $\chi_{\mathbb{C}G}$, χ_W and $\chi_U = \chi_{\mathbb{C}G} - \chi_W$.

We begin by writing down $\chi_{\mathbb{C}G}$: by question 7 on problem sheet 3, $\chi_{\mathbb{C}G}(g)$ is equal to the number of fixed points of the permutation g.

	e	(12)	(123)	$(12)(34)$	(1234)
size of conjugacy class	1	6	8	3	6
$\chi_{\mathbb{C}G}$	4	2	1	0	0
χ_W	1	1	1	1	1
χ_U	3	1	0	-1	-1

Now we can compute $\langle \chi_U, \chi_U \rangle$. Recall that

$$\langle \chi_U, \chi_U \rangle = \frac{1}{24} \sum_{g \in S_4} \chi_U(g)\overline{\chi_U(g)}.$$

We've written down the value of $\chi_U(g)$ for one g in each conjugacy class, so we need to multiply each of these terms by the size of the conjugacy class to get all the values. So we compute that

$$\frac{1}{24} \sum_{g \in S_4} \chi_U(g)\overline{\chi_U(g)} = \frac{1}{24}\left(3^2 + 6 \times (1)^2 + 8 \times (0)^2 + 3 \times (-1)^2 + 6 \times (-1)^2\right) = 1.$$

We conclude, using Corollary 3.3, that U is an irreducible representation of S_4.

Lecture 15

3.4. **Character tables.** The *character table* is a way of collecting information about all the irreducible characters of a finite group. Let G be a finite group. We have the set of irreducible characters of G: $\chi_1, \chi_2, \ldots, \chi_r$. We're also going to label the conjugacy classes of G: C_1, C_2, …, C_s. We have proved (Corollary 3.3) that $r \leq s$.

The character table for G is a table with columns indexed by the conjugacy classes $C_1, \ldots, C_s$ and rows indexed by the irreducible characters $\chi_1, \ldots, \chi_r$. The entry in the table in the χ_i row and C_j column is then given by $\chi_i(C_j) := \chi_i(g_j)$ where $g_j \in C_j$ is a representative for the conjugacy class C_j. So it looks like:

	C_1	C_2	$\cdots$	C_s
χ_1	$\chi_1(g_1)$	$\chi_1(g_2)$	$\cdots$	$\chi_1(g_s)$
$\vdots$				
χ_r	$\chi_r(g_1)$	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_s)$

Example 3.4. Let's start with a very small example. Let $G = C_2 = \{e, g\}$. There are two conjugacy classes $\{e\}$ and $\{g\}$, and two irreducible characters defined by $\chi_1(g) = 1$, $\chi_2(g) = -1$ and $\chi_i(e) = 1$. So the character table is:

	$\{e\}$	$\{g\}$
χ_1	1	1
χ_2	1	-1

Example 3.5. Now we're going to do a more interesting example, with $G = S_4$. Recall (from Problem sheet 2, Exercise 4) that there are two one-dimensional irreducible characters of G: namely the trivial character χ_{triv} which is defined by $\chi_{triv}(g) = 1$ for all $g \in G$ and the sign character χ_{sign} defined by $\chi_{sign}(g) = 1$ if g is an even permutation and $\chi_{sign}(g) = -1$ if g is an odd permutation. We have also found an irreducible three-dimensional representation U and computed its character. So we can start filling in the character table:

	e	(12)	(123)	(12)(34)	(1234)
size of conjugacy class	1	6	8	3	6
χ_{triv}	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_U	3	1	0	-1	-1

Note that I've added an extra row to tell us the size of the conjugacy classes, which is useful for computations. I've also labelled each conjugacy class by a representative element: for example, (12) denotes the conjugacy class of all transpositions.

So far we've found 3 irreducible characters. The sum of the squares of the dimensions is $1 + 1 + 9 = 11$ which is a lot less than $24 = |G|$ so we haven't found all of the irreducible characters yet!

What we've done so far allows us to write down another irrep: we can consider $U' := U \otimes V_{sign}$ where V_{sign} is the one dimensional rep with character χ_{sign}. The character $\chi_{U'}$ is equal to $\chi_U \chi_{sign}$ which is not equal to χ_U, and U' is also irreducible (see Problem Sheet 3: the tensor product of an irrep with a one-dimensional rep is irreducible; alternatively we can show that $\langle \chi_{U'}, \chi_{U'} \rangle = 1$). We conclude that we can put a new row in our character table:

	e	(12)	(123)	(12)(34)	(1234)
size of conjugacy class	1	6	8	3	6
χ_{triv}	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_U	3	1	0	-1	-1
$\chi_{U'}$	3	-1	0	-1	1

The sum of the squares of the dimensions of the irreps we've found so far is $1+1+9+9 = 20$, so we conclude that there must be one more irrep of dimension 2 (since we already found all the one-dimensional reps). Let's add this (currently mysterious) two-dimensional irrep into the character table:

	e	(12)	(123)	(12)(34)	(1234)
size of conjugacy class	1	6	8	3	6
χ_{triv}	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_U	3	1	0	-1	-1
$\chi_{U'}$	3	-1	0	-1	1
χ_V	2	?	?	?	?

Now the fun part is that we can work out what χ_V is, without constructing the representation V. We know that if χ is a irreducible character with $\chi \neq \chi_V$ then $\langle \chi_V, \chi \rangle = 0$. We also know that $V \otimes V_{sign}$ is a two-dimensional irrep, but since there is only one isomorphism class of two-dimensional irreps we get that $V \otimes V_{sign} \cong V$ and so $\chi_V \chi_{sign} = \chi_V$. This piece of information immediately tells us that $\chi_V(12) = \chi_V(1234) = 0$.

	e	(12)	(123)	(12)(34)	(1234)
size of conjugacy class	1	6	8	3	6
χ_{triv}	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_U	3	1	0	-1	-1
$\chi_{U'}$	3	-1	0	-1	1
χ_V	2	0	a	b	0

We've now labelled the two unknown entries in the character table by a, b. Now we use the fact that $\langle \chi_V, \chi_U \rangle = 0$. This says that $6 - 3b = 0$ which implies that $b = 2$.

	e	(12)	(123)	(12)(34)	(1234)
size of conjugacy class	1	6	8	3	6
χ_{triv}	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_U	3	1	0	-1	-1
$\chi_{U'}$	3	-1	0	-1	1
χ_V	2	0	a	2	0

Finally we use the fact that $\langle \chi_V, \chi_{triv} \rangle = 0$. This says that $2 + 8a + 6 = 0$ which implies that $a = -1$.

	e	(12)	(123)	(12)(34)	(1234)
size of conjugacy class	1	6	8	3	6
χ_{triv}	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_U	3	1	0	-1	-1
$\chi_{U'}$	3	-1	0	-1	1
χ_V	2	0	-1	2	0

So we've completed the character table!

Let's try and say a bit more about the representation V of S_4 whose character χ_V we just computed. Recall that we can work out $\ker \rho_V$ from the character χ_V: by Corollary 3.1 we know that $\ker \rho_V$ is given by those g such that $\chi_V(g) = \dim V = 2$. So the kernel is given by the normal subgroup $H = \{e, (12)(34), (13)(24), (14)(23)\}$ of S_4. This implies that the action of G on V is given by first mapping G to the quotient group G/H and then acting on V.

Definition 3.6. Suppose G is a finite group and $N \triangleleft G$ is a normal subgroup. Let $\overline{V}$ be a representation of the quotient group G/N. Then we get a representation V of G, called the *inflation* of $\overline{V}$ to G, with the same underlying vector space as the representation $\overline{V}$ and action of G defined by

$$\rho_V(g) = \rho_{\overline{V}}(gN).$$

Note that the inflated representation V is irreducible if and only if $\overline{V}$ is, because a G-stable subspace of V is the same thing as a G/N-stable subspace of $\overline{V}$.

Let's go back to our specific example with $G = S_4$ and consider the representation V. Since $H = \ker \rho_V$ the representation V is the inflation of a representation $\overline{V}$ of the quotient group G/H. In fact $\overline{V}$ is a two-dimensional irrep of S_4/H.

The quotient group S_4/H is isomorphic to S_3: consider the inclusion $S_3 \hookrightarrow S_4$ given by thinking of a permutation of $\{1,2,3\}$ as a permutation of $\{1,2,3,4\}$ which fixes 4. Composing with the quotient map $S_4 \to S_4/H$ gives a group homomorphism $S_3 \to S_4/H$ between two groups of order 6. Moreover this map is injective (since $S_3 \cap H = \{e\}$) and so S_3 is isomorphic to S_4/H.

We conclude that V is given by inflating the two-dimensional irrep of $S_3 \cong S_4/H$ to S_4.

Finally, we observe that for $G = S_4$ there are 5 conjugacy classes and 5 isomorphism classes of irreps.

Fact 3.1. For any finite group G, the number of conjugacy classes is equal to the number of isomorphism classes of irreps.

We will prove this fact in Chapter 4, but we're going to assume it for the rest of this Lecture 16 Chapter. This fact implies that the character table of a finite group is *square*.

3.5. **Key properties of the character table.** As usual we denote the irreducible characters by $\chi_1, \chi_2, \dots, \chi_r$.

(1) **The character table is square**. In other words, the number of conjugacy classes in a finite group G is equal to the number of irreducible characters.
(2) **Row orthogonality** (Corollary 3.3): $\langle \chi_i, \chi_j \rangle = 0$ if $i \neq j$ and $\langle \chi_i, \chi_i \rangle = 1$. Recalling the definition of the inner product $\langle , \rangle$ we have:
$$\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = 0, 1$$
if $i \neq j$, $i = j$ respectively.

Remark 3.4. Note that properties (1) and (2) recalled above imply that $\chi_1, \dots, \chi_r$ give an orthonormal basis for the vector space $C_{cl}(G)$ (with respect to the inner product $\langle , \rangle$). This is because $\chi_1, \dots, \chi_r$ are an orthonormal set of size r, with $r = \dim C_{cl}(G)$.

Moreover, if f is a class function we have
$$f = \sum_{i=1}^{r}\langle f, \chi_i\rangle \chi_i.$$

(3) **Column orthogonality** we refer to the content of the following Proposition as the column orthogonality relations for the character table:

Proposition 3.3. *Let $g \in G$, with conjugacy class $C(g)$. Then for any $h \in G$ we have*
$$\sum_{i=1}^{r}\overline{\chi_i(g)}\chi_i(h) = 0$$
if $h \notin C(g)$ and
$$\sum_{i=1}^{r}\overline{\chi_i(g)}\chi_i(h) = \frac{|G|}{|C(g)|}$$

if $h \in C(g)$.

Proof. We consider the class function δ_g defined by $\delta_g(h) = 1$ if $h \in C(g)$ and $\delta_g(h) = 0$ otherwise.

We have

$$\delta_g = \sum_{i=1}^{r} \langle \delta_g, \chi_i \rangle \chi_i.$$

Going back to the definition of the inner product we compute that

$$\langle \delta_g, \chi_i \rangle = \frac{|C(g)|}{|G|} \overline{\chi_i(g)}.$$

So we have

$$\delta_g = \sum_{i=1}^{r} \frac{|C(g)|}{|G|} \overline{\chi_i(g)} \chi_i.$$

Evaluating on h we get

$$\sum_{i=1}^{r} \frac{|C(g)|}{|G|} \overline{\chi_i(g)} \chi_i(h) = 0$$

if $h \notin C(g)$ and

$$\sum_{i=1}^{r} \frac{|C(g)|}{|G|} \overline{\chi_i(g)} \chi_i(h) = 1$$

if $h \in C(g)$. Multiplying by $\frac{|G|}{|C(g)|}$ gives the statement of the Proposition. $\square$

Example 3.6. As our first example of the column orthogonality relations we take $g = h = e$ in the statement of Proposition 3.3. Since $C(e) = \{e\}$, we get

$$\sum_{i=1}^{r} \overline{\chi_i(e)} \chi_i(e) = |G|.$$

Recalling that $\chi_i(e) = \dim V_i$ where V_i is an irrep with character χ_i, we get the familiar equation

$$\sum_{i=1}^{r} (\dim V_i)^2 = |G|.$$

Example 3.7. Here is an example of using the column orthogonality relations to fill in some missing entries in a character table. Let's take $G = D_8$, with generators s, t satisfying $s^4 = t^2 = e$, $tst = s^{-1}$. There are 5 conjugacy classes (and therefore 5 irreducible characters), and there are 4 isomorphism classes of one-dimensional representations. Using the equation

$$\sum_{i=1}^{r} (\dim V_i)^2 = |G|$$

gives that there is one remaining irreducible character, with dimension 2. So here is the character table, with 4 unknown entries:

	e	s	t	st	s^2
size of conjugacy class	1	2	2	2	1
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	-1	-1	1
χ_4	1	-1	-1	1	1
χ_5	2	a	b	c	d

We apply the column orthogonality relations to the e column and each of the other columns in turn. For example, taking $g = e$ and $h = s$ gives

$$0 = \sum_{i=1}^{5} \overline{\chi_i(e)}\chi_i(s) = 1 - 1 + 1 - 1 + 2a = 2a$$

which implies that $a = 0$. Similarly, we get $b = c = 0$ and $d = -2$.

3.6. **Row and column orthogonality.** Here is another way of formulating row and column orthogonality. We first define an $r \times r$ matrix B by

$$B_{ij} = \sqrt{\frac{|C_j|}{|G|}}\chi_i(g_j)$$

where the conjugacy classes in G are $C_1, C_2, \ldots, C_r$ with representatives $g_1, g_2, \ldots, g_r$.

Proposition 3.4. *The matrix B is* unitary. *In other words,*

$$B^{-1} = \overline{B}^t$$

Proof. It suffices to show that $B\overline{B}^t = I_r$ (the $r \times r$ identity matrix).[1]

Let's compute the (i, k) entry of the matrix $B\overline{B}^t$. We get

$$(B\overline{B}^t)_{ik} = \sum_{j=1}^{r} \chi_i(g_j)\overline{\chi_k(g_j)}\frac{|C_j|}{|G|} = \langle \chi_i, \chi_k \rangle$$

so we deduce from Corollary 3.3 that $B\overline{B}^t$ is the identity matrix. □

The proof of the above Proposition shows that the fact that B is unitary is equivalent to the row orthogonality. But if we multiply B and $\overline{B}^t$ the other way round, we see that the fact that B is unitary is also equivalent to the column orthogonality! So these facts are all equivalent.

We have

$$I_r = (\overline{B}^t B)_{ik} = \sqrt{\frac{|C(g_i)||C(g_k)|}{|G|^2}} \sum_{j=1}^{r} \overline{\chi_j(g_i)}\chi_j(g_k) = 0, 1$$

if $i \neq k$, $i = k$ respectively and this amounts to the statement of Proposition 3.3.

[1] Note that this implies that $\det(B)\det(\overline{B}^t) \neq 0$, so $\det(B) \neq 0$ and therefore B is invertible, with inverse $\overline{B}^t$. So we don't need to check that $\overline{B}^t B = I_r$ as well.

3.7. **What can you tell about a group from its character table?**

(1) Firstly, something negative: we can't identify a group (up to isomorphism) from its character table. For example, D_8 and the quaternion group Q_8 `http://en.wikipedia.org/wiki/Quaternion_group` have the same character table.
(2) We can work out the centre of a group from its character table. See Question 6 on Sheet 4.
(3) We can tell if a group is simple from its character table.
(4) We can find all the normal subgroups of a group from its character table.

We'll finish this chapter by explaining the final two points.

Definition 3.7. Let $\chi : G \to \mathbb{C}$ be a function. We define $\ker \chi = \{g \in G : \chi(g) = \chi(e)\}$.

Fact 3.2. If V is a rep of G, then $\ker \chi_V = \ker \rho_V$. See Corollary 3.1 and Sheet 4 Question 2.

Combining this with Sheet 3 question 8, we deduce that a group G is simple if and only if $\ker \chi = \{e\}$ for every non-trivial irreducible character χ.

Finally, we explain the point about normal subgroups:

Proposition 3.5. *Let G be a finite group and $H \triangleleft G$ a normal subgroup. Let $\chi_1, \chi_2, \dots \chi_r$ be the irreducible characters of G.*

(1) *There is a rep V of G such that* $\ker \rho_V = H$.
(2) *There is a subset $I \subset \{1, 2, \dots, r\}$ such that*
$$H = \bigcap_{i \in I} \ker \chi_i.$$

Proof. (1) For the first part we consider the regular representation $\mathbb{C}[G/H]$ of the quotient group. We let V be the inflation (see Definition 3.6) of $\mathbb{C}[G/H]$ to a rep of G. Then $\ker \rho_V = H$ (since $\mathbb{C}[G/H]$ is a faithful rep of G/H).

(2) Let V be as in part 1). We know that V is a direct sum of irreps, so we have
$$V \cong \bigoplus_{i=1}^{r} V_i^{\oplus m_i}$$
for integers $m_i \geq 0$. We let $I = \{i : m_i > 0\}$. Then we have
$$H = \ker \rho_V = \bigcap_{i \in I} \ker \rho_{V_i} = \bigcap_{i \in I} \ker \chi_i.$$
□ Lecture 17

4. Algebras and modules

In this chapter, our goal is to put the representation theory of finite groups in a slightly more abstract context. We also need to prove Fact 3.1.

A basic example of an algebra is the matrix algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries. Recall that this is a complex vector space (we can add matrices and scale by complex numbers). But we can also *multiply* matrices. In other words we have a map

$$\begin{aligned} m : M_n(\mathbb{C}) \times M_n(\mathbb{C}) &\to M_n(\mathbb{C}) \\ (M, N) &\mapsto M \cdot N \end{aligned}$$

given by matrix multiplication.

The multiplication map m has three key properties:

- m is *bilinear*: i.e. $m(\lambda_1 M_1 + \lambda_2 M_2, N) = \lambda_1 m(M_1, N) + \lambda_2 m(M_2, N)$ and similarly for $m(M, \lambda_1 N_1 + \lambda_2 N_2)$.
- m is *associative*: i.e. $L \cdot (M \cdot N) = (L \cdot M) \cdot N$.
- m is *unital*: i.e. there is an element, I_n (the identity matrix) such that $I_n \cdot M = M \cdot I_n = M$ for all $M \in M_n(\mathbb{C})$.

Our general definition of an algebra is a vector space with a multiplication satisfying the above three properties.

Definition 4.1. An *algebra* (over $\mathbb{C}$) is a vector space (over $\mathbb{C}$) equipped with a bilinear, associative, unital map $m : A \times A \to A$.

We write ab or $a \cdot b$ for $m(a, b)$. Suppose 1_A, $1'_A$ are two units for the multiplication on A. Then we have $1_A 1'_A = 1'_A$ (since 1_A is a unit) and $1_A 1'_A = 1_A$ (since $1'_A$ is a unit). So $1_A = 1'_A$, and a unit is unique. We write 1_A for the unit of A.

Alternative definition. If you've seen rings before: An algebra is a ring A, equipped with a ring homomorphism

$$\mu : \mathbb{C} \to Z(A)$$

where $Z(A)$ is the centre of A (recall that $Z(A) = \{z \in A : za = az \forall a \in A\}$).

To go from the alternative definition to the 'official definition' we define the multiplication map m to be given by ring multiplication, and the vector space structure on A to be given by ring addition and multiplication by $\mu(\lambda)$ for $\lambda \in C$.

Conversely, starting with the official definition, you get a ring structure on A by defining the ring multiplication to be given by the map m, and the ring homomorphism

$$\mu : \mathbb{C} \to Z(A)$$

is defined by $\mu(\lambda) = \lambda 1_A$.

Example 4.1.

1. The first example is the complex numbers $\mathbb{C}$ with its usual multiplication.
2. Let $A = \mathbb{C}[x] = \{a_0 + a_1 x + \cdots + a_n x^n : n \geq 0, a_i \in \mathbb{C}\}$ be polynomials in one variable over $\mathbb{C}$. Then the usual multiplication on A makes it into an algebra. Note that A is an infinite-dimensional vector space.
3. Let A be a two-dimensional vector space with basis $1, x$ and multiplication given by $1^2 = 1$, $1x = x1 = x$ and $x^2 = 0$. We can extend bilinearly to define the multiplication on two arbitrary elements of A. This defines an algebra, with unit 1. If you are familiar with rings and ideals, A is isomorphic to the quotient of $\mathbb{C}[x]$ by the ideal generated by x^2.

(4) Let V be a vector space. Let $A = \mathrm{Hom}_{\mathbb{C}}(V, V)$, with multiplication given by composition of linear maps. This is an algebra, and if V has dimension n then choosing a basis for V gives an isomorphism $A \cong M_n(\mathbb{C})$. We call $\mathrm{Hom}_{\mathbb{C}}(V, V)$ the *endomorphism algebra* of V.

(5) Finally, the most important example for this chapter is given by the *group algebra*. Let G be a (finite) group. We have already defined the vector space $\mathbb{C}G$ with basis $\{[g] : g \in G\}$. We define a multiplication on $\mathbb{C}G$ by setting $[g][h] = [gh]$ (and extending bilinearly). The unit for this multiplication is $[e]$. The multiplication is associative because the group multiplication is associative.

Let's do a couple of examples of group algebras:

Example 4.2. (1) Let $G = \{e\}$, the trivial group. Then $\mathbb{C}G$ is a one-dimensional vector space with basis $[e]$ and the multiplication is given by

$$(\lambda[e])(\mu[e]) = \lambda\mu[e].$$

(2) Let $G = C_2 = \{e, g\}$, the cyclic group of order two. The group algebra $\mathbb{C}C_2$ is two-dimensional with basis $[e], [g]$. The multiplication is given by

$$(\lambda_1[e] + \lambda_2[g])(\mu_1[e] + \mu_2[g]) = (\lambda_1\mu_1 + \lambda_2\mu_2)[e] + (\lambda_1\mu_2 + \lambda_2\mu_1)[g].$$

Definition 4.2. Let A, B be algebras. A linear map

$$f : A \to B$$

is an *algebra homomorphism* if

- $f(1_A) = 1_B$
- $f(a_1a_2) = f(a_1)f(a_2)$

An algebra homomorphism f is an *isomorphism* of algebras if f is an invertible linear map.

You can check that if f is an isomorphism of algebras then the inverse f^{-1} is also an algebra homomorphism (and hence an isomorphism of algebras).

Definition 4.3. Let A, B be algebras. We define a multiplication on the direct sum $A \oplus B$ by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2).$$

This makes $A \oplus B$ into an algebra, with unit $(1_A, 1_B)$.

Example 4.3. Consider the two algebras $\mathbb{C}C_2$ and $\mathbb{C} \oplus \mathbb{C}$, and the linear map defined by

$$\begin{aligned} f : \mathbb{C}C_2 &\to \mathbb{C} \oplus \mathbb{C} \\ [e] &\mapsto (1, 1) \\ [g] &\mapsto (1, -1) \end{aligned}$$

Then f is an algebra homomorphism: we have $f([e]) = (1, 1)$ and we need to check that $f(xy) = f(x)f(y)$ for $x, y \in \mathbb{C}C_2$. In fact it suffices to check for x, y basis elements (by bilinearing of multiplication and linearity of f), and this is easy to do. Moreover, f is invertible as a map of vector spaces (since it takes a basis for the left hand side to a basis for the right hand side). So f is an isomorphism of algebras.

The map f transports the slightly complicated multiplication law on $\mathbb{C}C_2$ (we wrote it out above), to the simpler multiplication law on the direct sum algebra $\mathbb{C} \oplus \mathbb{C}$. One of the main results of this chapter will be to generalise this picture to the group algebra $\mathbb{C}G$ for an arbitrary finite group G.

Lecture 18

4.1. **The opposite algebra.**

Definition 4.4. Let A be an algebra, with multiplication $m : A \times A \to A$. We define a new algebra A^{op}: it has the same underlying vector space as A but the multiplication is defined by

$$m^{op}(a, b) = m(b, a).$$

Remark 4.1. If A is commutative, i.e. $m(a, b) = m(b, a)$, then $A^{op} = A$.

Proposition 4.1. *Let G be a group. The algebras $\mathbb{C}G$ and $(\mathbb{C}G)^{op}$ are isomorphic, with an isomorphism given by*

$$I : \mathbb{C}G \to (\mathbb{C}G)^{op}$$
$$[g] \mapsto [g^{-1}].$$

Proof. I is an isomorphism of vector spaces, since it takes a basis to a basis. So we just need to check that I is an algebra homomorphism. It suffices to check that $m^{op}(I([g]), I([h])) = I([g][h])$. On the one hand, $m^{op}(I([g]), I([h])) = [h^{-1}][g^{-1}] = [h^{-1}g^{-1}]$. On the other, $I([g][h]) = I([gh]) = [(gh)^{-1}] = [h^{-1}g^{-1}]$. □

Remark 4.2. The above Proposition tells us that group algebras are special: there are algebras A such that A^{op} is not isomorphic to A.

4.2. **Modules.**

Definition 4.5. Let A be an algebra. A (left) A-module is a complex vector space M, together with an algebra homomorphism

$$\rho : A \to \mathrm{Hom}_{\mathbb{C}}(M, M).$$

Remark 4.3. If we have an A-module M, then we can define a map

$$A \times M \to M$$
$$(a, m) \mapsto a \cdot m := \rho(a)(m)$$

which allows us to multiply elements of M by elements of A. This multiplication satisfies, for $m, n \in M$ and $a, b \in A$:

(1) $a \cdot (m + n) = a \cdot m + a \cdot n$ (since $\rho(a)$ is linear)
(2) $(a + b) \cdot m = a \cdot m + b \cdot m$ (since ρ is linear)
(3) $(ab) \cdot m = a \cdot (b \cdot m)$ (since ρ is an algebra homomorphism)
(4) $1_A \cdot m = m$ (since $\rho(1_A) = \mathrm{id}_M$)

Proposition 4.2. *Let G be a (finite) group. A $\mathbb{C}G$-module is the same thing as a representation of G.*

Proof. We'll describe how to go from a rep of G to a $\mathbb{C}G$-module and vice versa. Let's start with a representation V of G. It comes equipped with a group homomorphism $\rho_V : G \to \mathrm{GL}(V)$. We make V into a $\mathbb{C}G$-module by defining:

$$\rho : \mathbb{C}G \to \mathrm{Hom}_{\mathbb{C}}(V, V)$$
$$\sum_{g\in G} \lambda_g [g] \mapsto \sum_{g\in G} \lambda_g \rho_V(g).$$

The fact that ρ is a algebra homomorphism follows easily from the fact that ρ_V is a group homomorphism.

Conversely, if we start with a $\mathbb{C}G$-module M, we have an algebra homomorphism

$$\rho : \mathbb{C}G \to \mathrm{Hom}_{\mathbb{C}}(M, M).$$

We define a group homomorphism

$$\rho_M : G \to \mathrm{GL}(M)$$
$$g \mapsto \rho([g]).$$

The fact that ρ_M is a homomorphism again follows easily from the fact that ρ is an algebra homomorphism. Note that $\rho([g])$ is invertible, with inverse $\rho([g^{-1}])$. □

Example 4.4. Consider the algebra $A = \mathbb{C} \oplus \mathbb{C}$. We showed that A is isomorphic to $\mathbb{C}C_2$, so understanding the modules for A is the same as understanding reps of C_2. Let's classify one dimensional A-modules. So we have a one-dimensional vector space M, and we need to define an algebra homomorphism

$$\rho : A \to \mathrm{Hom}_{\mathbb{C}}(M, M) = \mathbb{C}.$$

A has a basis given by $(0, 1)$ and $(1, 0)$, so ρ is determined by the values $\rho(1, 0)$ and $\rho(0, 1)$. Since ρ is an algebra homomorphism these values have to satisfy various relations.

We have

- $\rho(1, 0)\rho(1, 0) = \rho((1, 0)(1, 0)) = \rho(1, 0)$
- $\rho(0, 1)\rho(0, 1) = \rho((0, 1)(0, 1)) = \rho(0, 1)$
- $\rho(0, 1)\rho(1, 0) = \rho((0, 1)(1, 0)) = \rho(0, 0) = 0$
- $\rho(0, 1) + \rho(1, 0) = \rho(1, 1) = 1$.

Putting this all together, we see that there are two possibilities for ρ: we either have

$$\rho(0, 1) = 1, \rho(1, 0) = 0$$

or we have

$$\rho(0, 1) = 0, \rho(1, 0) = 1.$$

The fact that there are two possibilities correspond to the fact that C_2 has two isomorphism classes of one-dimensional reps.

Lecture 19

4.3. **Last lecture.** Last time, we defined A-modules for an algebra A and showed that if G is a finite group then a module for the group algebra $\mathbb{C}G$ is the same thing as a representation of G.

This tells us that modules for an algebra are generalisations of representations of a group. Our next task is to generalise some definitions and results from earlier in the course (for example, the definition of G-linear maps and Schur's lemma) from representations to modules.

4.4. **The regular representation.** Recall that if G is a (finite) group, we have defined a representation of G on the vector space $\mathbb{C}G$. Viewing this representation as a $\mathbb{C}G$-module, the action of $\mathbb{C}G$ is given by left multiplication. So we can generalise this example of a module to any algebra A:

For any algebra A we get an A-module with underlying vector space A, and action of A given by the algebra homomorphism

$$\begin{aligned} \rho : A &\to \mathrm{Hom}_{\mathbb{C}}(A, A) \\ a &\mapsto m_a \end{aligned}$$

where m_a denotes the map

$$\begin{aligned} m_a : A &\to A \\ b &\mapsto ab. \end{aligned}$$

In other words, we let A act on itself by left multiplication.

4.5. **Module homomorphisms.**

Definition 4.6. Let A be an algebra and let M, N be A-modules. A linear map

$$f : M \to N$$

is a *module homomorphism*, or *A-linear map* if $f(a \cdot m) = a \cdot f(m)$ for all $m \in M$.

If f is invertible then we say that it is an *isomorphism of A-modules.*

Remark. If f is an invertible linear map and a module homomorphism, then the inverse f^{-1} is also a module homomorphism (and therefore it's an isomorphism of A-modules as well).

Notation: if M and N are A-modules we write $\mathrm{Hom}_A(M, N) \subset \mathrm{Hom}_{\mathbb{C}}(M, N)$ for the vector subspace of A-linear maps from M to N.

Lemma 4.1. *Let $A = \mathbb{C}G$ and let M, N be A-modules (so we can also think of M, N as reps of G). Then $\mathrm{Hom}_A(M, N) = \mathrm{Hom}_G(M, N)$. In other words, a linear map $f : M \to N$ is G-linear if and only if it is A-linear.*

Proof. Suppose $f : M \to N$ is an A-linear map. Then, by definition, $f([g] \cdot m) = [g] \cdot f(m)$ for $g \in G$ and $m \in M$. But this says that $f(\rho_M(g)m) = \rho_N(g)f(m)$, where $\rho_M : G \to \mathrm{GL}(M)$ and $\rho_N : G \to \mathrm{GL}(N)$ define the action of G on the representations M, N. So f is G-linear.

Conversely, suppose $f : M \to N$ is a G-linear map. Then

$$f((\sum_{g\in G} \lambda_g[g]) \cdot m) = \sum \lambda_g f([g] \cdot m)$$

by linearity of f and now G-linearity tells us that this expression is equal to

$$\sum \lambda_g[g] \cdot f(m) = (\sum \lambda_g[g]) \cdot f(m).$$

We conclude that f is A-linear. $\square$

This Lemma tells us that the notion of a module homomorphism is a generalisation of the notion of G-linear maps.

4.6. **Submodules and simple modules.** Next, we generalise the notion of a subrepresentation.

Definition 4.6. Let A be an algebra and let M be an A-module. A *submodule* of M is a vector subspace $N \subset M$ such that $a \cdot n \in N$ for all $a \in A$ and $n \in N$. In other words, N is an A-stable subspace of M.

Exercise 4.1. Let $A = \mathbb{C}G$. Let M be an A-module. Show that a submodule of M is the same thing as a subrepresentation of M.

Exercise 4.2. Let M, N be A-modules and suppose $f : M \to N$ is an A-linear map. Then the kernel and image of f are submodules of M and N respectively.

Definition 4.7. If A is an algebra, M is an A-module and $m \in M$, then the *submodule generated by* m, denoted $A \cdot m$ is the subspace $\{a \cdot m : a \in A\} \subset M$. It is a submodule because $b \cdot (a \cdot m) = (ba) \cdot m \in Am$.

Definition 4.9. Let A be an algebra. A non-zero A-module M is *simple* if the only submodules of M are $\{0\}$ and M itself.

Exercise 4.3. Let $A = \mathbb{C}G$. Let M be an A-module. Show that M is a simple $\mathbb{C}G$-module if and only if M is an irreducible representation of G.

Lemma 4.2. *Let M be a simple A-module, and let m be a non-zero element of M. Then $A \cdot m = M$.*

Conversely, if M is a non-zero A-module, such that for every non-zero element m of M we have $A \cdot m = M$, then M is simple.

Proof. Since $A \cdot m$ is a submodule of M it is either zero or equal to M. But m is non-zero, so $A \cdot m$ is non-zero. We deduce that $A \cdot m = M$.

For the converse, suppose we have a non-zero submodule $N \subset M$. Let $m \in N$ be a non-zero element. Since N is a submodule, $A \cdot m \subset N$. But $A \cdot m = M$ by assumption, so $N = M$. Therefore the only submodules of M are $\{0\}$ and M, so M is simple. $\square$

Remark. Suppose A is an algebra which is finite dimensional as a complex vector space. Let M be a simple A-module. Then M is also finite dimensional as a complex vector space, since $M = A \cdot m$, which means that the map

$$A \to M$$
$$a \mapsto a \cdot m$$

is a surjective linear map from the finite dimensional vector space A to M.

Example 4.5.
(1) We know lots of examples of irreducible representations of finite groups G, so we immediately get a stock of examples of simple modules for algebras $\mathbb{C}G$.
(2) Let A be any algebra, and let M be an A-module which is one-dimensional as a complex vector space. Then M is a simple module.
(3) For $n \geq 1$, consider the matrix algebra $M_n(\mathbb{C})$. Consider the standard n-dimensional complex vector space $\mathbb{C}^n$. Thinking of the elements of $\mathbb{C}^n$ as column vectors, we can make $\mathbb{C}^n$ into an $M_n(\mathbb{C})$-module by letting a matrix act by matrix multiplication on a column vector. Another way of defining this module is that $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$ is isomorphic to $M_n(\mathbb{C})$ (take a linear map to the associated matrix with respect to the standard basis). So we have an algebra homomorphism (in fact it's an isomorphism)

$$\rho : M_n(\mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n).$$

$\mathbb{C}^n$ is an example of a simple $M_n(\mathbb{C})$-module: if we let $v \in \mathbb{C}^n$ be any non-zero vector, then the submodule $M_n(\mathbb{C}) \cdot v$ generated by v is equal to $\mathbb{C}^n$, because given a non-zero vector v and an arbitrary vector w we can always find a matrix M such that $Mv = w$. We deduce from Lemma 4.2 that $\mathbb{C}^n$ is a simple $M_n(\mathbb{C})$-module.
(4) Finally, let's consider the algebra (which we saw before) $A = \mathbb{C}[x]/x^2$ with basis $\{1, x\}$, unit 1 and multiplication determined by $x^2 = 0$.

Suppose M is a simple module for A. Since A is commutative, the subspace $x \cdot M := \{x \cdot m : m \in M\} \subset M$ is an A-submodule of M. As M is simple we have either $x \cdot M = 0$ or $x \cdot M = M$. If $x \cdot M = M$ then we can apply the argument again to show that $x^2 \cdot M = M$. But $x^2 = 0$, so $x^2 \cdot M = 0$. Since M is non-zero, this is a contradiction. So we have $x \cdot M = 0$. In other words multiplication by x is the zero map on M. So a general element of A, of the form $\lambda + \mu x$, must act on M as multiplication by λ. We conclude that the only simple modules for A are one-dimensional, with $\lambda + \mu x$ acting as multiplication by λ.

Lecture 20

Having defined simple modules, we can now give a generalisation of Schur's lemma.

Proposition 4.3.
(1) *Let A be an algebra and let M, N be simple A-modules. Suppose $f : M \to N$ is an A-linear map. Then $f = 0$ or f is an isomorphism.*
(2) *Suppose A is an algebra which is finite dimensional as a complex vector space. Let M be a simple A-module. Then an A-linear map*

$$f : M \to M$$

is equal to multiplication by a scalar $\lambda \in \mathbb{C}$.

Proof. The proofs are exactly the same as the proof of Schur's lemma which we gave for representations.

(1) $\ker f$ and $\operatorname{im} f$ are submodules of M, N respectively, so they are either $\{0\}$ or the whole module. The two possibilities we get are that $f = 0$ or $\ker f = \{0\}$, $\operatorname{im} f = N$, and in the latter case f is an isomorphism.
(2) Since A is finite dimensional over $\mathbb{C}$, the same is true for M (by the remark after Lemma 4.2). Now we argue as in the case of representations: f has an eigenvalue λ, and $f - \lambda \operatorname{id}_M$ has a non-zero kernel, so $f - \lambda \operatorname{id}_M = 0$ which implies that $f = \lambda \operatorname{id}_M$. □

4.7. **The centre of an algebra.**

Definition 4.8. Let A be an algebra. The *centre* $Z(A)$ of A is defined to be

$$Z(A) = \{z \in A : za = az \text{ for all } a \in A\}.$$

The centre $Z(A)$ is a commutative subalgebra of A.

Lemma 4.3. *Let A be a finite dimensional algebra and M a simple A-module. Let $z \in Z(A)$. Then there exists $\lambda_z \in \mathbb{C}$ (depending on z and M) such that*

$$z \cdot m = \lambda_z m \text{ for all } m \in M.$$

Proof. Multiplication by z defines an A-linear map from M to M. Now apply Schur's lemma, Proposition 4.3. □

Proposition 4.4. *Let $A = \mathbb{C}G$. Then*

$$Z(A) = \{\sum_{g \in G} f(g)[g] : f : G \to \mathbb{C} \text{ is a class function}\}$$

Proof. We have $z \in Z(A)$ if and only if $z[h] = [h]z$ for all $h \in G$. Suppose $z = \sum_{g \in G} \lambda_g [g]$. Then $z \in Z(A)$ if and only if $z = [h^{-1}]z[h]$ for all $h \in G$. In other words, we need

$$\sum_{g \in G} \lambda_g [g] = \sum_{g \in G} \lambda_g [h^{-1}gh].$$

Letting $g' = h^{-1}gh$, the right hand side is equal to

$$\sum_{g' \in G} \lambda_{hg'h^{-1}} [g'].$$

So $z \in Z(A)$ if and only if $\lambda_{hgh^{-1}} = \lambda_g$ for all $h \in G$. In other words, if and only if $g \mapsto \lambda_g$ is a class function. □

Now given a class function f on G, we can associate an element of the centre $\sum_{g \in G} \overline{f(g)}[g] \in Z(\mathbb{C}G)$. The following proposition tells us how this element acts on irreps of G.

Proposition 4.5. *Let V be an irrep of G and $f \in C_{cl}(G)$ a class function. Let*

$$z = \sum_{g \in G} \overline{f(g)}[g].$$

Then for $v \in V$ we have

$$z \cdot v = \frac{|G|}{\dim V} \langle \chi_V, f \rangle v.$$

Proof. We know by Schur's lemma that there exists $\lambda_z \in \mathbb{C}$ such that

$$z \cdot v = \lambda_z v$$

for $v \in V$. We just need to find the scalar λ_z. Consider the linear map

$$\times z : V \to V.$$

The trace of this map is equal to $\lambda_z \dim V$. On the other hand, this map is equal to $\sum_{g \in G} \overline{f(g)} \rho_V(g)$, which has trace $\sum_{g \in G} \overline{f(g)} \chi_V(g) = |G| \langle \chi_V, f \rangle$. Comparing the two expressions for the trace, we get the statement of the proposition. □

Lecture 21

We can now finally prove Fact 3.1.

Theorem 4.1. *Let G be a finite group. The number of irreducible characters of G is equal to the number of conjugacy classes in G.*

Proof. Write $\chi_1, \ldots, \chi_r$ for the irreducible characters of G. We know that $\chi_1, \ldots, \chi_r$ are an orthonormal subset of the vector space of class functions $C_{cl}(G)$ (with respect to the inner product $\langle , \rangle$). Since the dimension of $C_{cl}(G)$ is equal to the number of conjugacy classes in G, it suffices to prove that $\chi_1, \ldots, \chi_r$ are an *orthonormal basis* of $C_{cl}(G)$.

Suppose $f \in C_{cl}(G)$. We consider

$$f' := f - \sum_{i=1}^{r} \langle f, \chi_i \rangle \chi_i.$$

We have

$$\langle f', \chi_j \rangle = 0$$

for $j = 1, \ldots, r$, by orthogonality of the irreducible characters. To prove the theorem, it suffices to show that $f' = 0$.

We show that if $f' \in C_{cl}(G)$ satisfies

$$\langle f', \chi_j \rangle = 0$$

for $j = 1, \ldots, r$ then $f' = 0$. Indeed, let's consider $z \in Z(\mathbb{C}G)$ defined by

$$z := \sum_{g \in G} \overline{f'(g)} [g].$$

Let V be an irreducible representation of G. Multiplication by z defines a linear map of V, and Proposition 4.5 implies that this linear map is equal to zero, since $\langle \chi_V, f' \rangle = \overline{\langle f', \chi_V \rangle} = 0$. We have an isomorphism of $\mathbb{C}G$-modules

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} V_i^{\dim V_i}$$

and multiplication by z on each component V_i is equal to 0, so multiplication by z is equal to 0 on $\mathbb{C}G$. But $z[e] = z$, so we conclude that $z = 0$. This implies that $f' = 0$, and we are done. □

4.8. **Semisimple modules and algebras.** In the above proof, we used that fact that the regular representation decomposes as a direct sum of irreducible representations:

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} V_i^{\dim V_i}.$$

In the remainder of the course we are going to discuss algebras A which have the property that A (considered as an A-module) decomposes as a direct sum of simple A-modules. First we give the definition of a direct sum of A-modules: it's just like taking the direct sum of representations of a group G.

Definition 4.9. Let A be an algebra and let M, N be A-modules. Then the vector space $M \oplus N$ is naturally an A-module, with action of A given by

$$a \cdot (m, n) = (a \cdot m, a \cdot n).$$

Definition 4.10. An A-module M is *semisimple* if: for every submodule L of M there exists a submodule N of M with $L \cap N = \{0\}$ and $L + N = M$. In other words, there exists a submodule N which is complementary to L.

Example 4.6. A simple module M is semisimple: the only submodules of M are $\{0\}$ and M, and they have complementary submodules M and $\{0\}$.

If $A = \mathbb{C}G$, then every (finite dimensional) A-module is semisimple: this is Maschke's theorem.

Example 4.7. This is a non-example: Let $A = \mathbb{C}[x]/x^2$, the algebra with basis $1, x$ and multiplication determined by $x^2 = 0$ and 1 a unit. Consider the A-module $M = A$, and the submodule $L = \mathbb{C} \cdot x \subset A$ spanned by x. Exercise: L has no complementary submodule.

Lemma 4.4. *Let M be an A-module, with an isomorphism*

$$\alpha : M \cong S_1 \oplus S_2 \oplus \cdots \oplus S_n$$

to a direct sum of simple A-modules S_i.

If $L \subset M$ is a submodule, then there exists a subset $I \subset \{1, \dots, n\}$ such that $\alpha^{-1}(\bigoplus_{i \in I} S_i)$ is a complementary submodule to L.

Proof. For simplicity we assume that $M = S_1 \oplus S_2 \oplus \cdots \oplus S_n$. Taking into account the isomorphism α just adds some extra notation to the proof. Let $I \subset \{1, \dots, n\}$ be maximal such that $L \cap (\bigoplus_{i \in I} S_i) = \{0\}$. Set $X = L \oplus (\bigoplus_{i \in I} S_i)$. To show the lemma, it suffices to show that $X = M$. We do this by showing that for each $j = 1, \dots, n$, $S_j \subset X$. Since the subspaces $S_1, \dots, S_n$ span M, this shows that $X = M$.

Suppose $j \in I$. Then by definition of X, $S_j \subset X$. Suppose $j \notin I$. The vector space $S_j \cap X$ is a submodule of S_j, so it is either $\{0\}$ or S_j (S_j is simple). If it is equal to S_j, we are done. Suppose it is equal to $\{0\}$. Then $L \cap (\bigoplus_{i \in I \cup j} S_i) = \{0\}$, which contradicts the maximality of I. So we have proved the lemma. $\square$

Lemma 4.5. *Let M be an A-module, and assume M is a finite dimensional complex vector space. Then the following are equivalent:*

(1) *$M \cong \bigoplus_{i=1}^n S_i$ with the S_i simple A-modules*
(2) *M is a semisimple A-module*

Proof. We showed that 2) implies 1) in the setting of group representations (Corollary 2.2). Exactly the same proof works here.

Lemma 4.4 shows that 1) implies 2). □

Remark. The assumption that M is finite-dimensional is unnecessary, but then you have to allow infinite direct sums and work a little bit harder. See, for example, Chapter XVII of S. Lang 'Algebra' for this theory worked out in a more general setting. For simplicity I'm going to put finite-dimensional hypotheses in everywhere!

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Lemma 4.6. *Let A be an algebra, and suppose M is a finite dimensional, semisimple A-module. Let $L \subset M$ be a submodule. Then L is semisimple. Note that if N is a complementary submodule to L then N is also semisimple (since $N \subset M$ is also a submodule).*

Proof. Fix an isomorphism

$$\alpha : M \cong S_1 \oplus S_2 \oplus \cdots \oplus S_n$$

to a direct sum of simple A-modules S_i. Lemma 4.4 shows that there is a subset $I \subset \{1, \dots, n\}$ such that $N_0 := \alpha^{-1}(\bigoplus_{i \in I} S_i)$ is a complementary submodule to L. Write β for the A-linear map

$$\beta : M \to \bigoplus_{i \notin I} S_i$$

given by first applying α and then projecting to the components S_i for $i \notin I$. The kernel of this projection map is $\bigoplus_{i \in I} S_i$ (i.e. the things with component in S_i equal to 0 for $i \notin I$). So the kernel of β is equal to N_0. We deduce that the restriction of β to L,

$$\beta|_L : L \to \bigoplus_{i \notin I} S_i$$

is an isomorphism, since $L \cap N_0 = \{0\}$ and L has the same dimension as $\bigoplus_{i \notin I} S_i$. So L is semisimple.

Alternative proof using quotient modules: Let $L \subset M$ be a submodule. If N is a complementary submodule to L then the natural projection map $M \to M/L$ restricts to an A-linear isomorphism $N \cong M/L$. Lemma 4.4 shows that there exists a semisimple complementary submodule N_0 to L. So M/L is semisimple. So we have proved that any quotient module of M is semisimple. But L is isomorphic to M/N_0 (since L is complementary to N_0). So L is also semisimple. □

Definition 4.11. Let A be an algebra. A is a *semisimple algebra* if A is semisimple as an A-module (see section 4.4: A is an A-module under left multiplication).

Example 4.8. (1) Let G be a finite group. Then $\mathbb{C}G$ is a semisimple algebra.
(2) Consider the matrix algebra $M_n(\mathbb{C})$. Let $V = \mathbb{C}^n$ be the simple $M_n(\mathbb{C})$-module given by column vectors of length n. Then

$$M_n(\mathbb{C}) \cong \bigoplus_{i=1}^n V$$

where the isomorphism takes a matrix M to the columns of M.

(3) Suppose A and B are two semisimple algebras. Then $A \oplus B$ is a semisimple algebra.

Proposition 4.6. *Let A be a finite dimensional, semisimple algebra. Let M be a finite dimensional A-module. Then M is semisimple.*

Proof. Let $b_1, \dots, b_n$ be a basis for M. The map

$$p : A^{\oplus n} \to M$$
$$(a_1, \dots, a_n) \mapsto \sum_{i=1}^{n} a_i \cdot b_i$$

is a surjective A-linear map. Note that $A^{\oplus n}$ is a semisimple A-module, since it is a direct sum of simple A-modules. Let N be a complementary submodule to $\ker p \subset A^{\oplus n}$. Then the restriction of p to N gives an isomorphism

$$p|_N : N \to M.$$

So M is isomorphic to a submodule of a semisimple module, and is therefore semisimple (by Lemma 4.6).

Alternative proof using quotient modules: M is a quotient of the semisimple A-module $A^{\oplus n}$ so by the alternative proof of Lemma 4.6 M is a semisimple A-module. □

Theorem 4.2. *Let A be a finite dimensional, semisimple algebra. Then there are finitely many isomorphism classes of simple A-module. Let $S_1, \dots, S_r$ be a complete list of non-isomorphic simple A-modules, and set $d_i = \dim S_i$. Then there is an isomorphism of A-modules*

$$A \cong \bigoplus_{i=1}^{r} S_i^{\oplus d_i}.$$

Proof. The proof is exatly the same as for Corollary 2.6 (which is the case $A = \mathbb{C}G$). It's a good exercise to write out the proof for semisimple algebras! □

Lecture 23

Example 4.9. Let $A = M_n(\mathbb{C})$ and let V be the simple A-module given by column vectors of length n. We have seen that $A \cong V^{\oplus n}$. Combing this with Theorem 4.2 we deduce that every simple A-module is isomorphic to V. Moreover, if M is a finite-dimensional A-module, then $M \cong V^{\oplus r}$ where $r = \frac{\dim M}{n}$.

In fact, A-modules are basically the same as vector spaces (i.e. $\mathbb{C}$-modules). If we start from an A-module M, we can consider the vector space $\mathrm{Hom}_A(V, M)$.

$M \cong V^{\oplus r}$ then $\mathrm{Hom}_A(V, M) \cong \mathrm{Hom}_A(V, V)^{\oplus r} = \mathbb{C}^{\oplus r}$, since A-linear maps from V to V are multiplication by a scalar (by Schur's lemma). This gives a way to go from an A-module (of dimension rn) to a vector space of dimension r.

Conversely, if X is a vector space, then we can make the tensor product $V \otimes X$ into an A-module by defining

$$a \cdot (v \otimes x) = (a \cdot v) \otimes x$$

(and extending linearly to all of $V \otimes X$). This takes a vector space of dimension r to an A-module of dimension rn.

This equivalence between vector spaces and A-modules is an example of a *Morita equivalence.*

Next, we return to general finite-dimensional semisimple algebras, and we are going to use Theorem 4.2 to pin down the structure of A as an algebra.

Let A be a finite-dimensional semisimple algebra, and let $S_1, \ldots, S_r$ be a complete list of non-isomorphic simple A-modules. We write

$$\rho_i : A \to \mathrm{Hom}_{\mathbb{C}}(S_i, S_i)$$

for the algebra homomorphism giving the action of A on S_i. We get an algebra homomorphism

$$\begin{aligned} \rho : A &\to \bigoplus_{i=1}^{r} \mathrm{Hom}_{\mathbb{C}}(S_i, S_i) \\ a &\mapsto (\rho_i(a))_{i=1}^{r} \end{aligned}$$

by taking the direct sum of the algebra homomorphisms ρ_i.

Theorem 4.3 (Artin–Wedderburn)**.** *The homomorphism*

$$\rho : A \to \bigoplus_{i=1}^{r} \mathrm{Hom}_{\mathbb{C}}(S_i, S_i)$$

is an isomorphism *of algebras.*

Proof. First we note that by Theorem 4.2

$$\dim A = \sum_{i=1}^{r} (\dim S_i)^2 = \dim \left(\bigoplus_{i=1}^{r} \mathrm{Hom}_{\mathbb{C}}(S_i, S_i) \right)$$

so ρ is an algebra homomorphism between two algebras of the same (finite) dimension over $\mathbb{C}$. So to show ρ is an isomorphism, it suffices to check that is injective.

Suppose $\rho(a) = 0$. This means that $\rho_i(a) = 0$ for $i = 1, \ldots, r$ so multiplication by a gives the zero map on every simple A-module. Since A itself is isomorphic (as an A-module) to a direct sum of simple A-modules, this implies that multiplication by a gives the zero map on A. In particular, we have $a \cdot 1_A = 0$. But $a \cdot 1_A = a$, so $a = 0$. We conclude that ρ is injective, and therefore ρ is an isomorphism.

□

Corollary 4.1. *Let G be a finite group. Then we have an isomorphism of algebras*

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} M_{d_i}(\mathbb{C})$$

where $d_1, \ldots, d_r$ are the dimensions of the irreducible characters of G.

Proof. This is an immediate consequence of Theorem 4.3. □

Example 4.10. Let's take $G = S_3$ and apply the statement of the Theorem. We have 3 irreps of G, the trivial one-dimensional rep V_{triv}, the one-dimensional rep given by the sign homomorphism V_{sign} and an irreducible two-dimensional rep V. Let's fix the generators $s = (123)$ and $t = (23)$ of G. With respect to an appropriate basis, the matrix representation of V is given by

$$s \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$

and

$$t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\omega = e^{2\pi i/3}$. So the map ρ corresponds to the algebra homomorphism

$$\mathbb{C}S_3 \to \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$$

determined by

$$s \mapsto (1, 1, \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix})$$

and

$$t \mapsto (1, -1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

Theorem 4.3 tells us that this algebra homomorphism is in fact an isomorphism.

Lecture 24

4.9. **Some (non-examinable) results using algebraic integers.** We'll finish off the course by giving a couple of nice (non-examinable) applications of results we can prove using a tiny bit of algebraic number theory. The key definition is

Definition 4.12. Let $\alpha \in \mathbb{C}$. If α is a root of a monic polynomial

$$X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$$

with integer coefficients a_i, then we say that α is an algebraic integer.

Fact 4.1.
- If $\alpha \in \mathbb{Q}$ is an algebraic integer, then α is an integer.
- If α, β are algebraic integers, then $\alpha\beta$ and $\alpha + \beta$ are both algebraic integers.

Example 4.11. A root of unity is an algebraic integer. If G is a finite group, χ is a character, and $g \in G$, then $\chi(g)$ is a sum of roots of unity, and is therefore an algebraic integer.

Proposition 4.7. *Let χ be an irreducible character of a finite group G. Let $g \in G$ with conjugacy class $C(g)$. Then*

$$|C(g)| \frac{\chi(g)}{\chi(e)}$$

is an algebraic integer.

Proof. Let $V \subset \mathbb{C}G$ be a subrepresentation of $\mathbb{C}G$ with character χ. Let $z = \sum_{h \in C(g)}[h] \in \mathbb{C}G$. By Proposition 4.5 we have

$$z \cdot v = |C(g)|\frac{\chi(g)}{\chi(e)}v$$

for all $v \in V$. So $|C(g)|\frac{\chi(g)}{\chi(e)}$ is an eigenvalue of the multiplication by z map on $\mathbb{C}G$. This map has a matrix (with respect to the standard basis of $\mathbb{C}$ G) with integer entries, because multiplication by $[h]$ has a matrix with integer entries for each $h \in G$.

So the characteristic polynomial of this matrix is monic with integer coefficients, and $|C(g)|\frac{\chi(g)}{\chi(e)}$ is a root of this polynomial; hence it is an algebraic integer. ☐

Corollary 4.2. *Let V be an irrep of G. Then* $\dim V$ *divides* $|G|$.

Proof. Let $\chi = \chi_V$. We want to show that $\frac{|G|}{\chi(e)}$ is an integer. By row orthogonlaity, we have

$$\sum_{g \in G} \chi(g)\overline{\chi(g)} = |G|$$

so

$$\sum_{g \in G} \frac{\chi(g)}{\chi(e)}\overline{\chi(g)} = \frac{|G|}{\chi(e)}.$$

It suffices to show that the right hand side of this equation is an algebraic integer. We do this by showing that the left hand side is an algebraic integer. But the left hand side is a sum of things of the form

$$|C(g)|\frac{\chi(g)}{\chi(e)}\overline{\chi(g)} = |C(g)|\frac{\chi(g)}{\chi(e)}\chi(g^{-1})$$

and by Proposition 4.7 each of these terms is a product of two algebraic integers, and hence an algebraic integer. ☐

Finally we give an important application of representation theory to group theory.

Theorem 4.4. *Let p be a prime number, and let $d \geq 1$ be an integer. Suppose G is a finite group with a conjugacy class of size p^d. Then G is not simple.*

Proof. Fix $g \in C(g)$ with $C(g)$ the conjugacy class of size p^d. Let $\chi_1, \ldots, chi_r$ be the irreducible characters of G, with χ_1 the trivial character. Note that $g \neq e$ (its conjugacy class has size > 1). By column orthogonality for the g and e column of the character table, we have

$$1 + \sum_{i=2}^{r} \overline{\chi_i(e)}\chi_i(g) = 0.$$

Note that the $\chi_i(e)$ are integers, so the complex conjugation doesn't do anything. Dividing by p and rearranging we have

$$\sum_{i=2}^{r} \frac{\chi_i(e)}{p}\chi_i(g) = -\frac{1}{p}$$

and the right hand side is *not* an algebraic integer. We conclude that there is an $i \geq 2$ with $\frac{\chi_i(e)}{p}\chi_i(g)$ *not* an algebraic integer (otherwise the sum would be an algebraic integer). Fix this i, and let $\chi = \chi_i$. We have $\chi(g) \neq 0$ and $p \nmid \chi(e)$.

Since $\chi(e)$ and $|C(g)|$ are coprime, Bezout's lemma tells us that there are integers a, b such that $a\chi(e) + b|C(g)| = 1$. We deduce that

$$a|C(g)|\frac{\chi(g)}{\chi(e)} + b\chi(g) = \frac{\chi(g)}{\chi(e)}$$

and therefore Proposition 4.7 implies that $\frac{\chi(g)}{\chi(e)}$ is a non-zero (since $\chi(g) \neq 0$) algebraic integer.

Recall from Proposition 3.1 that $|\frac{\chi(g)}{\chi(e)}| \leq 1$. Let V be an irrep with character χ. We moreover recall that if $|\frac{\chi(g)}{\chi(e)}| = 1$ the $\rho_V(g) = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbb{C}^\times$. In fact, if $\frac{\chi(g)}{\chi(e)}$ is a non-zero algebraic integer then we do have $|\frac{\chi(g)}{\chi(e)}| = 1$: all the *conjugates* (see `en.wikipedia.org/wiki/Conjugate_element_(field_theory)`) of $\frac{\chi(g)}{\chi(e)}$ have norm ≤ 1 (because they are a sum of $\chi(e)$ roots of unity divided by $\chi(e)$) and the product of the conjugates is an integer.

So we conclude that $\rho_V(g) = \lambda\,\mathrm{id}_V$. We define $H \triangleleft G$ to be

$$H = \{h \in G : \rho_V(h) \in \mathbb{C}^\times\,\mathrm{id}_V\}.$$

Then H is a normal subgroup of G, and since $g \in H$, $H \neq \{e\}$.

Now suppose G is simple. Then $H = G$. But V is a non-trivial irrep, so ρ_V is faithful. But ρ_V is then an injective map from G to an Abelian group $\mathbb{C}^\times\,\mathrm{id}_V$. So G is Abelian. This contradicts the assumption that g has a conjugacy class with size > 1. We conclude that G is not simple. ☐

Corollary 4.3 (Burnside's theorem)**.** *Let p, q be prime numbers and let $a, b \in \mathbb{Z}_{\geq 0}$ be integers ≥ 0 with $a + b \geq 2$. Suppose G is a finite group with $|G| = p^a q^b$. Then G is not simple.*

Proof. First suppose $a = 0$ (the same argument works if $b = 0$). Then $|G| = q^b$ with $b \geq 2$. It is a standard group theory fact that the centre $Z(G)$ is non-trivial. For the proof of this fact, consider the conjugacy classes in G: the orders of the conjugacy classes divide $|G|$ (they are orbits under the conjugation action) and we have at least one conjugacy class of size 1 (the conjugacy class of e). Since the sum of the orders of the conjugacy classes is divisible by q, we must have other conjugacy classes of size 1. This says that the centre of G is non-trivial.

Now let $H \triangleleft G$ be the cyclic subgroup generated by an element of $Z(G)$ of order q. This is a non-trivial normal subgroup of G, so G is not simple.

Now we assume that $a, b \geq 1$. By Sylow theory, there is a subgroup Q of G with $|Q| = q^b$. By the preceding remarks, we know that $Z(Q) \neq \{e\}$. We let $g \in Z(Q)$ with $g \neq e$.

Consider the conjugacy class of g. By the orbit-stabiliser theorem, the size of this conjugacy class is equal to $|G|/|Cent_G(g)|$ where $Cent_G(g)$ is the centralizer of g in G.

Since $g \in Z(Q)$, Q is contained in the centralizer $Cent_G(g)$, so q^b divides $|Cent_G(g)|$. We conclude that $|C(g)| = p^r$ for some $r \geq 0$.

If $r = 0$, then $g \in Z(G)$. Proceeding as in the $a = 0$ case, we deduce that G is not simple.

If $r \geq 1$ then we can apply Theorem 4.4. $\square$

The end.