M3/4/5P12 PROBLEM SHEET ON MASTERY MATERIAL

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Exercise 1. Let G be a finite group, with $H \subset G$ a subgroup and let V be a representation of G. Suppose $W \subset \mathrm{Res}_H^G V$ is a subrepresentation of the restriction of V to a representation of H.

(a) Let $g \in G$, and consider the subspace $\rho_V(g)W \subset V$. Show that this subspace depends only on the left coset gH of g.

(b) If $C \in G/H$ is a left coset, write W_C for the subspace $\rho_V(g)W \subset V$, where $g \in C$. Fix a representative g_C for each left coset C and let $f : G \to W$ be an element of $\mathrm{Ind}_H^G W$. Show that

$$\rho_V(g_C)f(g_C^{-1}) \in W_C$$

is independent of the choice of coset representative g_C.

(c) Suppose the subspaces $\{W_C : C \in G/H\}$ together sum to give V and, the sum is direct. In other words, we have

$$V = \bigoplus_{C \in G/H} W_C.$$

Show that V is isomorphic to the induced representation $\mathrm{Ind}_H^G W$.

Hint: consider the map which takes $f \in \mathrm{Ind}_H^G W$ *to* $\sum_{C \in G/H} \rho_V(g_C)f(g_C^{-1})$.

This exercise shows that our definition of the induced representation gives something satisfying the (alternative) definition given by Serre in *Linear representations of finite groups*.

Solution 1. (a) Since W is a subrepresentation of $\mathrm{Res}_H^G V$, it is H-stable. So we have $\rho_V(h)W = W$ for all $h \in H$. So we have $\rho_V(gh)W = \rho_V(g)W$ for all $h \in H$, and therefore the subspace $\rho_V(g)W$ of V only depends on the coset gH.

(b) We need to show that $\rho_V(g_C h)f((g_C h)^{-1}) = \rho_V(g_C)f(g_C^{-1})$ for all $h \in H$. Since f is in $\mathrm{Ind}_H^G W$ we have $f((g_C h)^{-1}) = f(h^{-1}g_C^{-1}) = \rho_V(h)^{-1}f(g_C^{-1})$. So we get that $\rho_V(g_C h)f((g_C h)^{-1}) = \rho_V(g_C)\rho_V(h)\rho_V(h)^{-1}f(g_C^{-1}) = \rho_V(g_C)f(g_C^{-1})$ as desired.

(c) As suggested by the hint, we consider the map

$$\theta : \mathrm{Ind}_H^G W \to V$$

taking f to $\sum_{C \in G/H} \rho_V(g_C)f(g_C^{-1})$. The previous part shows that this map doesn't depend on the choice of coset representatives g_C. We need to check that θ is a G-linear isomorphism. Since $\dim V = [G : H] \dim W = \dim \mathrm{Ind}_H^G W$ it suffices to check that θ is G-linear and injective. For injectivity, suppose $\theta(f) = 0$. This implies that $f(g_C^{-1}) = 0$ for every left coset C, which implies that $f(g) = 0$ for every $g \in G$, as we can write $g = hg_C^{-1}$ for some left coset C and some $h \in H$.

It remains to show G-linearity. Suppose $g \in G$. Then $g \cdot f$ is the function which takes g' to $f(g'g)$. So we have

$$\theta(g \cdot f) = \sum_{C \in G/H} \rho_V(g_C)f(g_C^{-1}g) = \sum_{C \in G/H} \rho_V(g)\rho_V(g^{-1}g_C)f((g^{-1}g_C)^{-1}).$$

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As C runs over G/H, the elements $g^{-1}g_C$ run over a complete set of coset representatives for G/H. We deduce that

$$\theta(g \cdot f) = \sum_{C \in G/H} \rho_V(g)\rho_V(g^{-1}g_C)f((g^{-1}g_C)^{-1}) = \rho_V(g)\theta(f),$$

as desired.

Exercise 2. Let G be a finite group and suppose we have a subgroup $H \subset G$ and a subgroup $K \subset H$. Let W be a representation of K. Consider the representation

$$IW = \mathrm{Ind}_H^G(\mathrm{Ind}_K^H W).$$

(a) Show that if V is a representation of G, we have

$$\langle \chi_{IW}, \chi_V \rangle = \langle \chi_W, \chi_{\mathrm{Res}_K^G V} \rangle$$

(b) Show, using part a), that IW is isomorphic to $\mathrm{Ind}_K^G W$. *You can also try to show this directly, without using character theory.*

Solution 2. (a) We apply Frobenius reciprocity twice. First we have

$$\langle \chi_{IW}, \chi_V \rangle = \langle \chi_{\mathrm{Ind}_K^H W}, \chi_{\mathrm{Res}_H^G V} \rangle.$$

Applying Frobenius reciprocity once more gives the desired answer.

(b) We also have

$$\langle \chi_{\mathrm{Ind}_K^G W}, \chi_V \rangle = \langle \chi_W, \chi_{\mathrm{Res}_K^G V} \rangle$$

so we deduce that

$$\langle \chi_{\mathrm{Ind}_K^G W}, \chi_V \rangle = \langle \chi_{IW}, \chi_V \rangle$$

for all reps V of G. This implies that IW is isomorphic to $\mathrm{Ind}_K^G W$, since it must have the same decomposition into irreducibles (considering the inner product with χ_V where V is an irrep of G).

Exercise 3. Let $G = S_5$ and let $H = A_4$ be the subgroup of G given by even permutations of $\{1, 2, 3, 4\}$ which fix 5.

Let V be a three-dimensional irreducible representation of H (there's a unique such V up to isomorphism, see Question 3 on Sheet 4). Use Frobenius reciprocity to compute the decomposition of $\mathrm{Ind}_H^G V$ as a direct sum of irreducible representations of G (you can freely refer to the character table of S_5 — this is computed in Exercise 2 in the 'extra exercises' for Sheet 4).

Solution 3. To compute the decomposition of $\mathrm{Ind}_H^G V$ we need to compute $\langle \chi_{\mathrm{Ind}_H^G V}, \chi_W \rangle$ for each irrep W of G. Frobenius reciprocity says that $\langle \chi_{\mathrm{Ind}_H^G V}, \chi_W \rangle = \langle \chi_V, \chi_{\mathrm{Res}_H^G W} \rangle$, so we need to compute these inner products (between characters of H). We have

$$\langle \chi_V, \chi_{\mathrm{Res}_H^G W} \rangle = \frac{1}{12}\left(3\chi_W(e) - 3\chi_W((12)(34))\right).$$

So for example, if W is one of the irreps of S_5 with dimension 5, we get

$$\langle \chi_V, \chi_{\mathrm{Res}_H^G W} \rangle = \frac{1}{12}(15 - 3) = 1.$$

With the notation for the irreps of S_5 coming from Exercise 2 in the 'extra exercises' for Sheet 4 (apologies for the clash of notation, but I hope it's clear what is meant), the final answer is that $\mathrm{Ind}_H^G V$ is isomorphic to the representation $V \oplus V' \oplus (\wedge^2 V)^{\oplus 2} \oplus W \oplus W'$. You can check this has dimension $30 = 3 \cdot [S_5 : A_4]$, as it should.

Exercise 4. Suppose H is a subgroup of a finite group G, and let V be an irreducible representation of H. Let $\chi_1, \dots, \chi_r$ be the irreducible characters of G and suppose that

$$\chi_{\mathrm{Ind}_H^G V} = \sum_{i=1}^{r} d_i \chi_i.$$

Show that $\sum_{i=1}^r d_i^2 \leq [G:H]$.

Solution 4. We have

$$\sum_{i=1}^{r} d_i^2 = \langle \chi_{\mathrm{Ind}_H^G V}, \chi_{\mathrm{Ind}_H^G V} \rangle = \langle \chi_V, \chi_{\mathrm{Res}_H^G \mathrm{Ind}_H^G V} \rangle.$$

Since V is an irrep, this is the number of times V appears in the decomposition of $\mathrm{Res}_H^G \mathrm{Ind}_H^G V$ into irreps of H. Since the dimension of this representation is equal to $[G:H]\dim(V)$, we have V appearing $\leq [G:H]$ times in this decomposition (otherwise the dimension would be too big), which gives the desired inequality.

Exercise 5. Suppose H is a subgroup of a finite group G, and let V be a representation of H. Let $g \in G$ with conjugacy class $C(g)$. Suppose that $C(g) \cap H = D_1 \cup D_2 \cup \cdots \cup D_t$, where the D_i are conjugacy classes in H. Note that we can evaluate the character χ_V of V on each conjugacy class D_i, by defining $\chi_V(D_i) = \chi_V(h)$ for $h \in D_i$.

(a) Show that the character χ of $\mathrm{Ind}_H^G V$ is given by

$$\chi(g) = \frac{|G|}{|H|} \sum_{i=1}^{t} \frac{|D_i|}{|C(g)|} \chi_V(D_i)$$

(b) If V is the trivial one-dimensional representation, show that the character χ of $\mathrm{Ind}_H^G V$ is given by

$$\chi(g) = \frac{|G||C(g) \cap H|}{|H||C(g)|}$$

Solution 5. (a) Recall from the notes that the character of $\mathrm{Ind}_H^G V$ is given by

$$\chi(g) = \frac{1}{|H|} \sum_{g' \in G : g' g (g')^{-1} \in H} \chi_V(g' g (g')^{-1}).$$

Let's fix a conjugacy class D of H which is contained in $C(g)$, and count the number of g' such that $g' g (g')^{-1} \in D$. Let Σ_D denote the set of these g'. We have

$$\chi(g) = \frac{1}{|H|} \sum_{i=1}^{t} |\Sigma_{D_i}| \chi_V(D_i).$$

If we fix $d \in D$ and consider the g' such that $g' g (g')^{-1} = d$, then the number of such g' is equal to the size of the centralizer $Z_G(g)$ of g in G (the number is non-zero because d is conjugate to g). Indeed, if we have g_1, g_2 with $g_1 g g_1^{-1} = g_2 g g_2^{-1} = d$ then we have $g_2^{-1} g_1 \in Z_G(g)$ and so $g_1 = g_2 z$ for some $z \in Z_G(g)$. The size of $Z_G(g)$ is $|G|/|C(g)|$ (by the orbit-stabilizer theorem). Adding the contributions for all d, we see that $|\Sigma_D| = |G||D|/|C(g)|$.

(b) In this case we have $\chi_V(D_i) = 1$ for all i. So we get

$$\chi(g) = \frac{|G|}{|H||C(g)|} \sum_{i=1}^{t} |D_i| = \frac{|G||C(g) \cap H|}{|H||C(g)|}.$$