M3/4/5P12 PROBLEM SHEET 4 (EXTRA EXERCISES)

Please send any corrections or queries to `j.newton@imperial.ac.uk`. These additional exercises work out the character tables of S_5 and A_5. They are fairly long/tricky but I've included them because it's good to see the computation of these character tables!

Exercise 1. Let $G = S_n$ and set $\Omega = \{1, \dots, n\}$. Recall that we have an n-dimensional rep $\mathbb{C}\Omega$ of S_n, with a one-dimensional subrepresentation spanned by $\sum_{i=1}^n [i]$. Let $V \subset \mathbb{C}\Omega$ be a complementary subrepresentation to this one-dimensional rep. The aim of this exercise is to show that V is irreducible.

For $g \in S_n$ write $Fix_\Omega(g)$ for the subset $\{i \in \Omega : gi = i\} \subset \Omega$. Recall that

$$\chi_{\mathbb{C}\Omega}(g) = |Fix_\Omega(g)|.$$

(See Exercise 7 on Problem Sheet 3).

(1) For $i, j \in \Omega$ define $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$. Show that

$$|Fix_\Omega(g)| = \sum_{i=1}^n \delta_{gi,i}.$$

(2) Show that

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = \frac{1}{n!} \sum_{g \in S_n} \left(\sum_{i=1}^n \delta_{gi,i} \right)^2.$$

(3) By multiplying out the square in the previous equation, and reordering the sum, show that

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = \frac{1}{n!} \sum_{i=1}^n \sum_{j=1}^n \sum_{g \in S_n} \delta_{gi,i} \delta_{gj,j}.$$

(4) Show that if $i = j$ then

$$\sum_{g \in S_n} \delta_{gi,i} \delta_{gj,j} = (n-1)!$$

(5) Show that if $i \neq j$ then

$$\sum_{g \in S_n} \delta_{gi,i} \delta_{gj,j} = (n-2)!$$

(6) Deduce that

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = 2.$$

(7) Finally, show that

$$\langle \chi_V, \chi_V \rangle = 1$$

and deduce that V is an irreducible representation of S_n.

Solution 1. (1) The sum $\sum_{i=1}^n \delta_{gi,i}$ simply counds the number of i such that $gi = i$. This is the number of fixed points of g in Ω.

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(2) We have $\chi_{\mathbb{C}\Omega}(g) = \sum_{i=1}^{n} \delta_{gi,i}$ by part (1). By definition

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = \frac{1}{|S_n|} \sum_{g \in S_n} \chi_{\mathbb{C}\Omega}(g) \overline{\chi_{\mathbb{C}\Omega}(g)}$$

and substituting in $\chi_{\mathbb{C}\Omega}(g) = \sum_{i=1}^{n} \delta_{gi,i}$ we get the desired answer.

(3) I think I gave too much of a hint for this part! You multiply out the square, reorder the sum and then you get what is written.

(4) If $i = j$ then the sum $\sum_{g \in S_n} \delta_{gi,i} \delta_{gj,j}$ counts the number of g which fix the single element i. There are $n-1$ other elements of Ω, and we can have any permutation of these, so we get $(n-1)!$ elements g in total.

(5) If $i \neq j$ then the sum counts the number of g which fix the two distinct elements i, j. There are $n-2$ other elements, giving $(n-2)!$ permutations in total.

(6) Combining parts (3), (4) and (5) we get that

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = \frac{1}{n!} \left(\sum_{i=j} (n-1)! + \sum_{i \neq j} (n-2)! \right).$$

There are n terms in the first sum and $n(n-1)$ in the second sum, so we get

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = \frac{1}{n!} (n! + n!) = 2.$$

(7) We have $\mathbb{C}\Omega \cong V \oplus U$ where U is isomorphic to the trivial rep. So

$$\langle \chi_{\mathbb{C}\Omega}, \chi_{\mathbb{C}\Omega} \rangle = \langle \chi_V, \chi_V \rangle + \langle \chi_V, \chi_U \rangle + \langle \chi_U, \chi_V \rangle + \langle \chi_U, \chi_U \rangle.$$

Since $\langle \chi_U, \chi_U \rangle = 1$ and $\langle \chi_U, \chi_V \rangle = \langle \chi_V, \chi_U \rangle = \dim \mathrm{Hom}_{S_n}(U, V) \geq 0$ and $\langle \chi_V, \chi_V \rangle$ is a positive integer, we conclude that $\langle \chi_V, \chi_V \rangle = 1$ (and $\langle \chi_U, \chi_V \rangle = 0$). So V is an irrep.

Exercise 2. There are 7 conjugacy classes in S_5, with representatives

$$e, (12), (123), (1234), (12345), (12)(34), (12)(345)$$

and sizes

$$1, 10, 20, 30, 24, 15, 20$$

respectively.

Recall that the one-dimensional characters of S_5 are given by χ_{triv} and χ_{sign}.

(1) In the previous exercise we found a four-dimensional irrep V for S_5. Write down the character χ_V of V and show that $V' := V \otimes V_{sign}$ gives a four-dimensional irrep which is not isomorphic to V.

(2) Using Exercise 6 on Problem Sheet 3, find the character of $\wedge^2 V$ and show that $\wedge^2 V$ is irreducible.

(3) Again using Exercise 6 on Problem Sheet 3, find the character of $S^2 V$ and show that $S^2 V \cong V_{triv} \oplus V \oplus W$, where W is a representation of dimension 5. Show moreover that W is irreducible, and $W' := W \otimes V_{sign}$ is another, non-isomorphic, irrep of dimension 5.

We have now found all the irreps of S_5, and their characters. There are 7 isomorphism classes of irreps: $V_{triv}, V_{sign}, V, V', \wedge^2 V, W$ and W'.

Solution 2. (1) We have $\chi_V(g) = |Fix_\Omega(g)| - 1$. So we get

	e	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
χ_V	4	2	1	0	-1	0	-1

Since (12) is odd and $\chi_V(12)$ is non-zero, we get that $\chi_{V'} \neq \chi_V$ so V' is a four-dimensional irrep, not isomorphic to V.

(2) We have

$$\chi_{\wedge^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2}.$$

So we compute

	e	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
$\chi_{\wedge^2 V}$	6	0	0	0	1	-2	0

We then get that $\langle \chi_{\wedge^2 V}, \chi_{\wedge^2 V}\rangle = 1$, so $\wedge^2 V$ is irreducible.

(3) We have

$$\chi_{S^2 V} + \chi_{\wedge^2 V} = \chi_{V \otimes V} = (\chi_V)^2.$$

So we compute

	e	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
$\chi_{S^2 V}$	10	4	1	0	0	2	1

We then get that $\langle \chi_{S^2 V}, \chi_{triv}\rangle = \langle \chi_{S^2 V}, \chi_V = 1$, so $S^2 V \cong V_{triv} \oplus V \oplus W$ where W has dimension 5. We now compute $\chi_W = \chi_{S^2 V} - \chi_{triv} - \chi_V$:

	e	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
χ_W	5	1	-1	-1	0	1	1

We can now check that $\langle \chi_W, \chi_W\rangle = 1$, so W is irreducible. Finally, since $\chi_W(12)$ is non-zero, we get that W' is non-isomorphic to W.

Exercise 3. There are 5 conjugacy classes in A_5, with representatives

$$e, (123), (12345), (13452), (12)(34)$$

and sizes

$$1, 20, 12, 12, 15$$

respectively.

(1) Show that the representations V and W of the previous exercise restrict to irreducible representations of A_5 (which we still call V, W).

(2) Show that the representation $\wedge^2 V$ restricts to a representation X of A_5 whose character χ_X has $\langle \chi_X, \chi_X\rangle = 2$. Deduce that X decomposes as a direct sum of two non-isomorphic irreducible representations Y, Z of A_5.

(3) Deduce that the complete list of irreps (up to isomorphism) of A_5 is given by V_{triv}, V, W, Y, Z, and show that $\dim(Y)^2 + \dim(Z)^2 = 18$, hence $\dim(Y) = \dim(Z) = 3$.

Here's the character table so far (note that we know $\chi_Y + \chi_Z$ because we know χ_X):

	e	(123)	(12345)	(13452)	(12)(34)
χ_{triv}	1	1	1	1	1
χ_V	4	1	-1	-1	0
χ_W	5	-1	0	0	1
χ_Y	3	a	b	c	d
χ_Z	3	$-a$	$1-b$	$1-c$	$-2-d$

(4) Show that if V is a rep of A_5 then $\overline{\chi_V(g)} = \chi_V(g)$. *Hint: If $g \in A_5$ then g^{-1} is conjugate to g, so $\chi_V(g^{-1}) = \chi_V(g)$.*

(5) Using the column orthogonality relations

$$\sum_{i=1}^{r} |\chi_i(g)|^2 = |G|/|C(g)|$$

where $C(g)$ is the conjugacy class of g, show that $a = 0$, $d = -1$ and b, c are both solutions to the quadratic equation $x^2 - x - 1 = 0$.

(6) Conclude that the character table of A_5 is given by

	e	(123)	(12345)	(13452)	(12)(34)
χ_{triv}	1	1	1	1	1
χ_V	4	1	-1	-1	0
χ_W	5	-1	0	0	1
χ_Y	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
χ_Z	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1

Solution 3. (1) We just need to restrict the characters χ_V, χ_W to A_5 and check that the inner products with themselves are equal to 1.

(2) Again we just compute that inner product of χ_X with itself. If we write

$$X \cong \bigoplus_{i=1}^{r} V_i^{\oplus m_i}$$

with V_i non-isomorphic irreps, then we get that $\langle \chi_X, \chi_X \rangle = \sum_{i=1}^r m_i^2$, so we must have two non-zero m_i's, both equal to 1. This says that X decomposes as a direct sum of two non-isomorphic irreps.

(3) We have produced 5 non-isomorphic irreps, and there are 5 conjugacy classes in A_5, so these are all the irreps. The formula $|G| = \sum_{i=1}^r d_i^2$ now says that

$$60 = 1 + 16 + 25 + \dim(Y)^2 + \dim(Z)^2$$

so we get $18 = \dim(Y)^2 + \dim(Z)^2$. Therefore the only possibility for the dimensions is $\dim(Y) = \dim(Z) = 3$.

(4) Using the hint, we see that $\chi_V(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ as required.

(5) Part two implies that the unknows a, b, c, d are all real. Now column orthogonality for the first column says $3 + 2a^2 = 3$ and for the fourth says $2 + 2d^2 + 4d + 4 = 4$ which says that $d^2 + 2d + 1 = (d + 1)^2 = 0$. Finally, we get the same relation for the second and third column, and it looks like $2 + 2b^2 - 2b + 1 = 5$ which gives $b^2 - b - 1 = 0$, and similarly for c.

(6) We didn't distinguish between Y and Z yet, and b, c must be the two distinct roots of $x^2 - x - 1$ in some order. We can therefore choose b to be the root $\frac{1+\sqrt{5}}{2}$, and this gives the character table written out in the question.