M3/4/5P12 PROBLEM SHEET 4

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Exercise 1. Let G be a finite group, and $g \in G$ an element of order 2. Let V be a representation of G. Show that $\chi_V(g)$ is an integer and that

$$\chi_V(g) \equiv \dim V \pmod 2.$$

Hint: recall that $\chi_V(g)$ is a sum of eigenvalues of $\rho_V(g)$.

Solution 1. Since g is an element of order 2, we have $\rho_V(g)^2 = \mathrm{id}_V$. So the eigenvalues of $\rho_V(g)$ are equal to 1 or -1. Therefore $\chi_V(g)$ (which is the sum of these eigenvalues) is a sum of $\dim V$ integers equal to 1 or -1. Since $-1 \equiv 1 \pmod 2$ we get that $\chi_V(g)$ is an integer which is $\equiv \dim(V) \pmod 2$.

Exercise 2. Let $\chi : G \to \mathbb{C}$ be a function. Define $\ker \chi$ by

$$\ker \chi = \{g \in G : \chi(g) = \chi(e)\}.$$

Now suppose V is a representation of G, with $\rho_V : G \to \mathrm{GL}(V)$ the homomorphism giving the action of G on V, and χ_V the character of V.

Show that $\ker \chi_V = \ker \rho_V$.

Solution 2. Corollary 3.1 in the lecture notes says that $\rho_V(g) = \mathrm{id}_V$ if and only if $\chi_V(g) = \dim(V)$. Since $\dim(V) = \chi_V(e)$, this says that the kernel of ρ_V is equal to the kernel of χ_V.

Exercise 3. In this exercise we are going to work out the character table of $A_4 \subset S_4$, the group of even permutations of $\{1, 2, 3, 4\}$. There are 4 conjugacy classes in A_4, with representatives $e, (123), (132), (12)(34)$ and sizes $1, 4, 4, 3$ respectively.

(1) Show that A_4 has an irreducible representation U of dimension 3 with character given by

$$\chi_U(e) = 3, \chi_U(123) = \chi_U(132) = 0, \chi_U((12)(34)) = -1.$$

Hint: restrict a three-dimensional irrep of S_4 to the subgroup A_4

(2) Show that A_4 has three isomorphism classes of irreps of dimension 1, one isomorphism class of irreps of dimension 3 and these are all the irreps.

You've now shown that the character table of A_4 looks like:

	e	(123)	(132)	(12)(34)
χ_{triv}	1	1	1	1
χ_U	3	0	0	-1
χ_3	1	?	?	?
χ_4	1	?	?	?

(3) Show that $\chi_3((12)(34)) = \chi_4((12)(34)) = 1$. *Hint: use the fact that $\langle \chi, \chi' \rangle = 0$ if $\chi \neq \chi'$ are distinct irreducible characters.*

(4) Fill in the rest of the character table. *Hint: if χ is the character of a one-dimensional rep then $\chi(123)^3 = \chi(132)^3 = 1$. We also know that $\langle \chi_3, \chi_{triv} \rangle = \langle \chi_4, \chi_{triv} \rangle = 0$.*

Date: Sunday 6th March, 2016.

(5) (More advanced question) Show that the representations with characters χ_3 and χ_4 are obtained by inflating representations of a quotient of A_4 which is isomorphic to the cyclic group C_3.

Solution 3. (1) We know that S_4 has a three-dimensional irreducible representation, and we wrote down its character in lectures: it's given by $\chi(g) =$ (the number of fixed points of the permutation g)-1. Restricting this rep of S_4 to A_4 gives a rep of A_4, and the character is given by restricting the character of the rep of S_4 to A_4.

(2) A_4 has size 12, and we just showed there's an irrep of dimension 3. The formula $12 = \sum_{i=1}^{r} d_i^2$ tells us that there must be 3 one-dimensional irreps and these give all the isomorphism classes of irreps.

(3) Since $\langle \chi_3, \chi_U \rangle = \langle \chi_4, \chi_U \rangle = 0$, we get that $\chi_3(e)\cdot 3 + 3\cdot\chi_3((12)(34))\cdot(-1) = 0$ and similarly for χ_4, which implies that $\chi_3((12)(34)) = \chi_4((12)(34)) = 1$.

(4) (123) and (132) have order 3, so the one-dimensional characters χ_3, χ_4 give cube roots of unity on (123) and (132). Since $(132) = (123)^{-1}$ we also get that $\chi(132) = \overline{\chi(123)}$. We conclude that the character table looks like

	e	(123)	(132)	(12)(34)
χ_{triv}	1	1	1	1
χ_U	3	0	0	-1
χ_3	1	a	$\overline{a}$	1
χ_4	1	b	$\overline{b}$	1

where a, b are third roots of unity. Since $\chi_3 \neq \chi_4$ we have $a \neq b$ and we also have $a \neq 1$, $b \neq 1$, since the characters are not trivial. So we can, without loss of generality, assume that $a = e^{2\pi i/3}$ and $b = e^{4\pi i/3}$. So we finally get

	e	(123)	(132)	(12)(34)
χ_{triv}	1	1	1	1
χ_U	3	0	0	-1
χ_3	1	ω	ω^{-1}	1
χ_4	1	ω^{-1}	ω	1

where $\omega = e^{2\pi i/3}$.

(5) We can read of the kernel of the representations with characters χ_3 and χ_4 from the kernel of the characters (see exercise 2). We have $\ker \chi_3 = \ker \chi_4 = \{e, (12)(34), (13)(24), (14)(23)\} = H$, the normal subgroup of order 4. The quotient group A_4/H is isomorphic to the cyclic group of order 3 (because this is the only group of order 3, up to isomorphism!). So these representations are inflated from this quotient group (and they are inflated from the two non-trivial characters...).

Exercise 4. (1) Let U be the three-dimensional irrep of A_4 found in the previous exercise. Find the decomposition of $U \otimes U$ into irreducibles.

(2) Let V be the two-dimensional irrep of S_4 found in lectures. Find the decomposition into irreducibles of the restriction of V to a representation of A_4.

Solution 4. (1) The character $\chi_{U\otimes U}$ is given by

	e	(123)	(132)	(12)(34)
$\chi_U \cdot \chi_U$	9	0	0	1

Now we need to compute $\langle \chi_U \cdot \chi_U, \chi_i \rangle$ for each irreducible character χ_i. We have $\langle \chi_U \cdot \chi_U, \chi_{triv} \rangle = \langle \chi_U \cdot \chi_U, \chi_3 \rangle = \langle \chi_U \cdot \chi_U, \chi_4 \rangle = \frac{1}{12}(9+3) = 1$ and $\langle \chi_U \cdot \chi_U, \chi_U \rangle = \frac{1}{12}(27-3) = 2$, so we get

$$U \otimes U \cong U^{\oplus 2} \oplus V_{triv} \oplus V_3 \oplus V_4.$$

(2) Let's recall the character of V (restricted to A_4):

	e	(123)	(132)	(12)(34)
χ_V	2	-1	-1	2

so we have $\chi_V = \chi_3 + \chi_4$, and $V \cong V_3 \oplus V_4$.

Exercise 5. Let G be a finite group such that every irrep of G is one-dimensional. Show that G is Abelian. *Hint: how many conjugacy classes does G have?*

Solution 5. The number of irreducible characters equals the number of conjugacy classes (actually for this question it's enough to know that the number of irreducible characters is $\leq$ the number of conjugacy classes). Since every irrep has dimension 1, the formula $|G| = \sum_{i=1}^{r} d_i^2 = \sum_{i=1}^{r} 1$ tells us that the number of irreducible characters is equal to $|G|$. So there (at least) $|G|$ conjugacy classes in G. This is only possible if every conjugacy class has size one. In other words, we have $hgh^{-1} = g$ for all $g, h \in G$, so $hg = gh$ and G is Abelian.

Exercise 6. Let G be a finite group, with irreducible characters $\chi_1, \chi_2 \ldots, \chi_r$. Fix an element $g \in G$. Show that g is in the centre of G (i.e. $gh = hg$ for all $h \in G$) if and only if

$$\sum_{i=1}^{r} \chi_i(g)\overline{\chi_i(g)} = |G|.$$

Solution 6. Recall the column orthogonality relation:

$$\sum_{i=1}^{r} \chi_i(g)\overline{\chi_i(g)} = \frac{|G|}{|C(g)|}.$$

So we have

$$\sum_{i=1}^{r} \chi_i(g)\overline{\chi_i(g)} = |G|$$

if and only if $|C(g)| = 1$, which happens if and only if $hgh^{-1} = g$ for all $h \in G$, i.e. if g is in the centre of G.

Exercise 7. (1) Write down the character table of S_3.

(2) Consider the class function $\phi : S_3 \to \mathbb{C}$ defined by $\phi(e) = 4, \phi(12) = 0, \phi(123) = -5$. Write ϕ as a linear combination of irreducible characters of S_3.

(3) Is ϕ the character of a representation of S_3?

Solution 7. (1) Here is the character table:

	e	(12)	(123)
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_V	2	0	-1

(2) We have $\langle \phi, \chi_{triv} \rangle = \frac{1}{6}(4 + 2 \cdot (-5)) = -1$, $\langle \phi, \chi_{sign} \rangle = \frac{1}{6}(4 + 2 \cdot (-5)) = -1$, $\langle \phi, \chi_V \rangle = \frac{1}{6}(8 + 2 \cdot (-5) \cdot (-1)) = 3$, so

$$\phi = 3\chi_V - \chi_{triv} - \chi_{sign}.$$

(3) Since the irreducible characters are linearly independent in the vector space of class functions, there is a unique way to write ϕ as a linear combination of irreducible characters. Some of the coefficients are negative, but a character of a representation has all its coefficients non-negative (they are dimensions of vector spaces). So ϕ is not a character of a rep.