M3/4/5P12 PROBLEM SHEET 3

Please send any corrections or queries to `j.newton@imperial.ac.uk`.

Exercise 1. (1) Let V be a finite dimensional vector space. Consider the map

$$\alpha : V \to (V^*)^*$$

defined by letting $\alpha(v)$ be the linear map

$$\alpha(v) : V^* \to \mathbb{C}$$

given by $\alpha(v)(\delta) = \delta(v)$, for $\delta \in V^*$. Show that this map is an isomorphism of vector spaces.

(2) If V is a representation of G, show that α is a G-linear isomorphism.

Solution 1. (1) It's not hard to check that α is a linear map. Since V and $(V^*)^*$ have the same dimension, it suffices to show that α is injective. If $\alpha(v) = 0$ then $\delta(v) = 0$ for all $\delta \in V^*$. This implies that $v = 0$: if $v \neq 0$ then we can find a basis $v = v_1, v_2, \dots v_n$ for V and consider $\delta_v \in V^*$ given by $\delta_v(v_i) = 1$ if $i = 1$ and $\delta_v(v_i) = 0$ if $i > 1$.

(2) We need to check that $\rho_{V^{**}}(g) \circ \alpha = \alpha \circ \rho_V(g)$. We have $\alpha(\rho_V(g)v) \in V^{**}$ given by

$$\alpha(\rho_V(g)v)\delta = \delta(\rho_V(g)v) = (\rho_{V^*}(g^{-1})\delta)(v) = \alpha(v)(\rho_{V^*}(g^{-1})\delta)$$

On the other hand $\rho_{V^{**}}(g)\alpha(v)$ is given by the map

$$\delta \mapsto \alpha(v)(\rho_{V^*}(g^{-1})\delta)$$

so we have the desired equality.

Exercise 2. Let G be a finite group and consider the regular representation $\mathbb{C}G$. Show that the dual $(\mathbb{C}G)^*$ is isomorphic to $\mathbb{C}G$ as a representation of G.

Solution 2. Since $\mathbb{C}G \cong \bigoplus_W W^{\oplus \dim W}$ where the sum is over distinct isomorphism classes of irreps, the dual $(\mathbb{C}G)^*$ is isomorphic to

$$\bigoplus_W (W^*)^{\oplus \dim W} = \bigoplus_W (W^*)^{\oplus \dim(W^*)}.$$

Since W^* is irreducible and $(W^*)^* \cong W$ the map $W \mapsto W^*$ is just a permutation (of order 2) of the set of isomorphism classes of irreducible representations, and so we have

$$\bigoplus_W (W^*)^{\oplus \dim(W^*)} \cong \bigoplus_W W^{\oplus \dim W}.$$

As a consequence, we have $(\mathbb{C}G)^* \cong \mathbb{C}G$.

Alternatively, you can check directly that if $\delta_[g]$ is the dual basis to the usual basis $\{[g] : g \in G\}$ then the map

$$[g] \mapsto \delta_{[g]}$$

is a G-linear isomorphism between $\mathbb{C}G$ and $(\mathbb{C}G)^*$.

Yet another alternative is to consider the matrix representation: the matrices M given by $\rho_{\mathbb{C}G}(g)$ with respect to the usual basis are *permutation matrices*: every row and column contains a single 1 with zeros everywhere else, and you can check

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that MM^t is the identity matrix, so $M = M^{-t}$ and the dual matrix representation is the same as the original matrix representation.

The next two exercises explain a way to think about tensor products of vector spaces without fixing bases. They are not essential for the course.

Exercise 3. Let V, W and X be complex vector spaces. A map

$$f : V \times W \to X$$

is called bilinear if it is linear in each variable separately. That is, $f(av_1 + bv_2, w) = af(v_1, w) + bf(v_2, w)$ and $f(v, aw_1 + bw_2) = af(v, w_1) + bf(v, w_2)$ for $a, b \in \mathbb{C}$.

(1) Show that the map $\pi : V \times W \to V \otimes W$ which takes (v, w) to $v \otimes w$ is a bilinear map. *Note that we have implicitly fixed bases of V, W to define $V \otimes W$.*

(2) Show that for every bilinear map $f : V \times W \to X$ there is a unique linear map $h : V \otimes W \to X$ such that $f = h \circ \pi$.

(3) *This part is trickier* Suppose $\pi' : V \times W \to U$ is a bilinear map, and for every bilinear map $f : V \times W \to X$ there is a unique linear map $h : U \to X$ such that $f = h \circ \pi'$. Show that there is a unique isomorphism $i : U \to V \otimes W$ such that $i \circ \pi' = \pi$.

Remarks: Part (2) of the exercise says that tensor products are a way to turn bilinear maps into linear maps.

We can also use this exercise to give an alternative (basis-independent) definition of the tensor product. We say that a vector space U, together with a bilinear map $\pi : V \times W \to U$ 'is a tensor product' of V and W if for every bilinear map $f : V \times W \to X$ there is a unique linear map $h : V \otimes W \to X$ such that $f = h \circ \pi$. Part (2) says that a tensor product of V and W exists (it's the tensor product $V \otimes W$ we have already defined with a chosen basis of V and W).

Part (3) says that a tensor product of V and W is unique up to unique isomorphism, so to all intents and purposes any two tensor products of V and W are the same mathematical object.

Solution 3. As usual we fix bases $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$ for V and W.

(1) I'll just write out the proof of linearity in the first variable. We have $\pi(av_1 + bv_2, w) = (av_1 + bv_2) \otimes w$. If we write $v_1 = \sum_i \lambda_i a_i$ and $v_2 = \sum_i \mu_i a_i$ then we have

$$av_1 + bv_2 = \sum_i (a\lambda_i + b\mu_i) a_i$$

and then it's straightforward to check that

$$(av_1 + bv_2) \otimes w = \sum_i (a\lambda_i + b\mu_i)(a_i \otimes w) = a(v_1 \otimes w) + b(v_2 \otimes w).$$

(2) First we check uniqueness: if $f = h \circ \pi$ and $f = h' \circ \pi$ then we have $f(v, w) = h(v \otimes w) = h'(v \otimes w)$. In particular, h and h' are equal on a basis for $V \otimes W$, so $h = h;$.

Next we check existence: we define h by $h(a_i \otimes b_j) = f(a_i, b_j)$. We now need to check that $f = h \circ \pi$. In other words, we need to check that $h(v \otimes w) = f(v, w)$ for all $v \in V, w \in W$. Writing $v = \sum_i \lambda_i a_i$ and $w = \sum_j \mu_j b_j$ we have by definition that $v \otimes w = \sum_{i,j} \lambda_i \mu_j (a_i \otimes b_j)$. By the definition of h we have

$$h(v \otimes w) = \sum_{i,j} \lambda_i \mu_j f(a_i, b_j).$$

By bilinearity of f this is also equal to $f(v, w)$, so we are done.

(3) We set $X = V \otimes W$ and $f = \pi$. Then there is a unique linear map $h : U \to V \otimes W$ such that $\pi = h \circ \pi'$. We now need to check that h is an isomorphism. We apply part (2) to the bilinear map π'. This tells us that there is a unique linear map $h' : V \otimes W \to U$ such that $\pi' = h' \circ \pi$. If we consider the map $L = h' \circ h$ we get a linear map $L : U \to U$ such that $L \circ \pi' = \pi'$. The map id_U also satisfies this, so by our uniqueness assumption we have $L = \mathrm{id}_U$. Applying the same argument to $h \circ h'$ we get that h and h' are inverse to each other, so h is an isomorphism.

Exercise 4. Let V, W be two vectors spaces, with bases A and B. Let $\mathbb{C}[V \times W]$ be the (infinite dimensional) complex vector space with basis given by symbols $\{v * w : v \in V, w \in W\}$. Define a linear map $\mathbb{C}[V \times W] \to V \otimes W$ by taking $v * w$ to $v \otimes w$.

(1) Let $E \subset \mathbb{C}[V \times W]$ be the subspace spanned by the elements

$$(av_1 + bv_2) * w - a(v_1 * w) - b(v_2 * w), \quad v * (aw_1 + bw_2) - a(v * w_1) - b(v * w_2)$$

with $a, b \in \mathbb{C}$, $v_i, v \in V$ and $w_i, w \in W$.

Note that a map of sets $f : V \times W \to X$ gives a linear map

$$F : \mathbb{C}[V \times W] \to X$$

defined by $F(v*w) = f(v, w)$. Show that f is bilinear if and only if $F(u) = 0$ for all $u \in E$.

(2) Show that the map $(v, w) \mapsto v * w$ defines a bilinear map from $V \times W$ to the quotient vector space $\mathbb{C}[V \times W]/E$.

(3) Using Exercise 3, show that there is a unique isomorphism

$$i : \mathbb{C}[V \times W]/E \to V \otimes W$$

satisfying $i(v * w + E) = v \otimes w$.

Solution 4. (1) The definition of bilinearity unwinds to the statement that $F(u) = 0$ for $u \in E$.

(2) Let $X = \mathbb{C}[V \times W]/E$ and let $f : V \times W \to X$ be the map which takes (v, w) to $v*w+E$. The map F is then the quotient map $\mathbb{C}[V \times W] \to \mathbb{C}[V \times W]/E$. Since this map is zero on E, applying part (1) we conclude that f is bilinear.

(3) We apply part (3) of Exercise (3), with $U = \mathbb{C}[V \times W]/E$.

Exercise 5. (1) Let V and W be representations of G and suppose W has dimension one. Show that $V \otimes W$ is irreducible if and only if V is irreducible.

(2) Let V and W be representations of G. Show that $V \otimes W$ is isomorphic as a representation of G to $W \otimes V$.

Solution 5. (1) Suppose that V is reducible, and $U \subset V$ is a proper subspace. Then we can think of $U \otimes W$ as a subspace of $V \otimes W$: let $\{a_1, \ldots, a_l\}$ be a basis of U and extend to a basis $\{a_1, \ldots, a_m\}$ for V. Let b be a basis vector for W. Then the vectors $a_i \otimes b$ for $1 \leq i \leq l$ span a subrepresentation of $V \otimes W$ which is isomorphic to $U \otimes W$. So $V \otimes W$ is also reducible.

Conversely if $V \otimes W$ is reducible, it has a subrepresentation U'. Let $x_1, \ldots x_l$ be a basis for U'. Since every element of $V \otimes W$ is of the form $v \otimes b$ with $v \in V$ we can write $x_i = v_i \otimes b$. Now it is straightforward to check that the v_i span a proper subrepresentation of V.

Alternatively we can just apply the first half of our answer again, since $(V \otimes W) \otimes W^* \cong V$.

(2) The map which takes $a_i \otimes b_j$ to $b_j \otimes a_i$ defines a G-linear isomorphism between $V \otimes W$ and $W \otimes V$.

Exercise 6. (1) Let V be a representation of G and consider the map $f : V \otimes V \to V \otimes V$ given by $f(v_1 \otimes v_2) = v_2 \otimes v_1$. Show that f is a G-linear map.

(2) Define S^2V to be the subspace of $x \in V \otimes V$ such that $f(x) = x$. Define $\wedge^2 V$ to be the subspace of $x \in V \otimes V$ such that $f(x) = -x$. Show that S^2V and $\wedge^2 V$ are subrepresentations of $V \otimes V$ and $V \otimes V \cong S^2V \oplus \wedge^2 V$.

(3) Show that $(1/2)(f + \mathrm{id}_V)$ is a projection with image S^2V and $(1/2)(f - \mathrm{id}_V)$ is a projection with image $\wedge^2 V$.

(4) Show that if $A = \{a_1, \ldots, a_n\}$ is a basis of V, then $\{a_i \otimes a_j - a_j \otimes a_i : i < j\}$ is a basis of $\wedge^2 V$. What are the dimensions of S^2V and $\wedge^2 V$ in terms of $\dim(V) = n$? Can you find a basis for S^2V?

(5) Suppose $g \in G$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues (with multiplicity) of $\rho_V(g)$. Show that the eigenvalues of $\rho_{\wedge^2 V}(g)$ are $\{\lambda_i \lambda_j : i < j\}$.

(6) Show that the characters $\chi_{\wedge^2 V}$ and $\chi_{S^2 V}$ are given by

$$\chi_{\wedge^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2} \qquad \chi_{S^2 V}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}.$$

Solution 6. (1) It's not hard to check that f is a linear map. We have

$$f(\rho_{V \otimes V}(g) v_1 \otimes v_2) = f(\rho_V(g) v_1 \otimes \rho_V(g) v_2) = \rho_V(g) v_2 \otimes \rho_V(g) v_1 = \rho_{V \otimes V}(g) f(v_1 \otimes v_2)$$

so f is G-linear.

(2) Note that $f \circ f = \mathrm{id}_{V \otimes V}$. So f is diagonalisable with eigenvalues 1 and -1. Since $S^2V = \ker(f - \mathrm{id}_{V \otimes V})$ and $\wedge^2 V = \ker(f + \mathrm{id}_{V \otimes V})$ are the kernels of G-linear maps, they are subrepresentations. Since they also give the two eigenspaces of f we have

$$V \otimes V \cong S^2 V \oplus \wedge^2 V.$$

(3) Since $(1/2)(f + \mathrm{id}_V)$ is zero on $\wedge^2 V$ and the identity on S^2V, it is a projection with image S^2V. The same argument holds for $(1/2)(f - \mathrm{id}_V)$.

(4) Applying $(1/2)(f - \mathrm{id}_V)$ to the basis vectors $a_i \otimes a_j$, we see that $\wedge^2 V$ is spanned by vectors $a_i \otimes a_j - a_j \otimes a_i$. So it is spanned by the vectors $\{a_i \otimes a_j - a_j \otimes a_i : i < j\}$. These vectors are linearly independent since $\sum_{i<j} \lambda_{i,j}(a_i \otimes a_j - a_j \otimes a_i) = 0$ implies that each $\lambda_{i,j}$ is equal to zero (it is a sum of distinct basis vectors).

We conclude that the dimension of $\wedge^2 V$ is $(1/2)n(n-1)$. Since $\dim(V \otimes V) = n^2$ we also get that $\dim S^2 V = (1/2)n(n+1)$. A basis for S^2V is given by $\{a_i \otimes a_j + a_j \otimes a_i : i \leq j\}$.

(5) Letting A be a basis of eigenvectors for $\rho_V(g)$, we get that $\{a_i \otimes a_j - a_j \otimes a_i : i < j\}$ is a basis for $\wedge^2 V$ and the eigenvalue of $\rho_{\wedge^2 V}(g)$ on $a_i \otimes a_j = a_j \otimes a_i$ is $\lambda_i \lambda_j$, which gives the desired (multi)set of eigenvalues.

(6) We have $\chi_V(g) = \lambda_1 + \cdots + \lambda_n$ and

$$\chi_V(g)^2 = (\lambda_1 + \cdots + \lambda_n)^2 = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i \neq j} \lambda_i \lambda_j.$$

So

$$\chi_V(g)^2 - \chi_V(g^2) = \sum_{i \neq j} \lambda_i \lambda_j = 2\chi_{\wedge^2 V}(g)$$

which gives the desired formular for $\chi_{\wedge^2 V}$. So get $\chi_{S^2 V}$ we use the fact that

$$\chi_{S^2 V} + \chi_{\wedge^2 V} = \chi_{V \otimes V} = \chi_V \chi_V.$$

Exercise 7. Let G be a group acting on a finite set Ω. Recall that we have defined a representation $\mathbb{C}\Omega$ of G. Show that the character $\chi_{\mathbb{C}\Omega}$ satisfies: $\chi_{\mathbb{C}\Omega}(g)$ is equal to the number of fixed points for g in Ω.

Solution 7. Think about the matrix for $\rho_{\mathbb{C}\Omega}(g)$ with respect to the basis $\{[\omega] : \omega \in \Omega\}$. The diagonal entry of this matrix corresponding to the basis element $[\omega]$ is equal to 1 if $g\omega = \omega$ and equal to 0 otherwise. So the trace is equal to the number of ω with $g\omega = \omega$, or in other words the number of fixed points of g.

Exercise 8.
(1) Let G be a finite group. Show that if G is simple (i.e. G is non-trivial and the only normal subgroups of G are $\{e\}$ and G) then a representation of G is either trivial or faithful.
(2) Suppose every non-trivial irreducible representation of a finite group G is faithful. Show that G is a simple group. *Hint: if G is not simple then there is a normal subgroup N of G such that G/N is simple.*

Solution 8.
(1) If a representation V is not faithful then the kernel of $\rho_V : G \to \mathrm{GL}(V)$ is a normal subgroup of G which is not equal to $\{e\}$. Since G is simple, this subgroup must be equal to G. So V is trivial.
(2) Suppose G is not simple. Then there is a normal subgroup N of G with G/N simple. There is a non-trivial irreducible representation V of G/N (since G/N is non-trivial). By restriction along the map $G \to G/N$ we get a non-trivial representation of G on the vector space V: it is irreducible since if $U \subset V$ is a G-stable subspace it is also a G/N stable subspace. So by our assumption V is a faithful representation of G. But N acts trivially on V, which is a contradiction.