M3/4/5P12 PROBLEM SHEET 2

Please send any corrections or queries to `j.newton@imperial.ac.uk`.

Exercise 1. Let V, ρ_V and W, ρ_W be representations of a group G with dimension m and n respectively. Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ be bases for V and W.

Let $A \oplus B$ be the basis for $V \oplus W$ given by $(a_1, 0), \ldots, (a_m, 0), (0, b_1), \ldots, (0, b_n)$. Describe the matrix representation

$$(\rho_V \oplus \rho_W)_{A\oplus B} : G \to \mathrm{GL}_{m+n}(\mathbb{C})$$

in terms of the matrix representations $(\rho_V)_A$ and $(\rho_W)_B$.

Solution 1. The matrix

$$[\rho_V(g) \oplus \rho_W(g)]_{A\oplus B}$$

is block diagonal, with the first $m \times m$ block given by $[\rho_V(g)]_A$ and the second, $n \times n$ block given by $[\rho_W(g)]_B$. So the matrix representation $(\rho_V \oplus \rho_W)_{A\oplus B}$ is given by these block diagonal matrices.

Another way of saying this is that taking a pair

$$(M, N) \in \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$$

to the block diagonal matrix with first block M and second block N gives a homomorphism

$$I : \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{m+n}(\mathbb{C}).$$

The matrix representations $r = (\rho_V)_A$ and $s = (\rho_W)_B$ give a homomorphism

$$r \times s : G \to \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$$

and the composition $I \circ (r \times s) : G \to \mathrm{GL}_{m+n}(\mathbb{C})$ is the matrix representation $(\rho_V \oplus \rho_W)_{A\oplus B}$.

Exercise 2. Let V and W be representations of a group G. Recall that $\mathrm{Hom}_{\mathbb{C}}(V, W)$ denotes the complex vector space of linear maps from V to W.

(1) Let $g \in G$ act on $\mathrm{Hom}_{\mathbb{C}}(V, W)$ by taking a linear map $f : V \to W$ to the linear map

$$g \cdot f : v \mapsto \rho_W(g) f(\rho_V(g^{-1})v).$$

Show that this defines a representation of G on $\mathrm{Hom}_{\mathbb{C}}(V, W)$. What is the dimension of this representation, in terms of the dimensions of V and W?

(2) Show that the invariants $\mathrm{Hom}_{\mathbb{C}}(V, W)^G$ in this representation of G are the G-linear maps $\mathrm{Hom}_G(V, W)$.

Solution 2. (1) First we check that we have defined a representation. One way of writing the definition is that it takes f to the composition of maps

$$\rho_W(g) \circ f \circ \rho_V(g^{-1}).$$

So it takes a linear combination $\lambda f_1 + \mu f_2$ to

$$\rho_W(g) \circ (\lambda f_1 + \mu f_2) \circ \rho_V(g^{-1}) = \lambda \rho_W(g) \circ f_1 \circ \rho_V(g^{-1}) + \mu \rho_W(g) \circ f_2 \circ \rho_V(g^{-1}).$$

Date: Tuesday 9^{th} February, 2016.

So the map $\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g)$ which sends f to $\rho_W(g) \circ f \circ \rho_V(g^{-1})$ gives a linear map from $\mathrm{Hom}_{\mathbb{C}}(V,W)$ to $\mathrm{Hom}_{\mathbb{C}}(V,W)$. We can also see that $\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(e)$ is the identity.

Now we need to check that

$$\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(gh) = \rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g) \circ \rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(h)$$

for $g, h \in G$. Note that this also shows that the linear map $\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g)$ is invertible, since an inverse is given by $\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g^{-1})$. For $f \in \mathrm{Hom}_{\mathbb{C}}(V,W)$ We have

$$\begin{aligned}
\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g) \circ \rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(h)f &= \rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g) \left(\rho_W(h) \circ f \circ \rho_V(h^{-1})\right) \\
&= \rho_W(g)\rho_W(h) \circ f \circ \rho_V(h^{-1})\rho_V(g^{-1}) \\
&= \rho_W(gh) \circ f \circ \rho_V(h^{-1}g^{-1}) = \rho_W(gh) \circ f \circ \rho_V((gh)^{-1}) \\
&= \rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(gh)f
\end{aligned}$$

so we have indeed defined a representation.

The dimension of $\mathrm{Hom}_{\mathbb{C}}(V,W)$ is equal to $\dim(V) \cdot \dim(W)$. One way to show this is to observe that, fixing a basis A for V (with dimension m) and a basis B for W (with dimension n), the map

$$f \mapsto [f]_{A,B}$$

gives an isomorphism of complex vector space

$$\mathrm{Hom}_{\mathbb{C}}(V,W) \cong M_{n\times m}(\mathbb{C})$$

between $\mathrm{Hom}_{\mathbb{C}}(V,W)$ and $n \times m$ matrices. $M_{n\times m}(\mathbb{C})$ has dimension mn: a basis is given by the matrices with 1 in one entry and zeroes everywhere else.

(2) We need to show that $f \in \mathrm{Hom}_{\mathbb{C}}(V,W)$ is G-linear if and only if

$$\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g)f = f$$

for all g in G.

We have

$$\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g)f = \rho_W(g) \circ f \circ \rho_V(g^{-1})$$

so composing with the invertible map $\rho_V(g)$ on both sides we see that

$$\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g)f = f$$

if and only if

$$\rho_W(g) \circ f = f \circ \rho_V(g).$$

This holds for all g if and only if f is G-linear (by definition), so we have shown that f is G-linear if and only if

$$\rho_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g)f = f$$

for all g in G, as desired.

Exercise 3. Recall that we proved in lectures that if U is a subrepresentation of a representation V of a finite group, then there exists a complementary subrepresentation $W \subset V$ with $V \cong U \oplus W$ (Maschke's theorem).

Prove by induction that if V is a representation of a finite group G, then V is isomorphic to a direct sum

$$V_1 \oplus V_2 \oplus \cdots \oplus V_d$$

with each V_i an irreducible representation of G. *A proof is written in the typed lecture notes if you get stuck!.*

Solution 3. We induct on the dimension of V. It is obvious that a one-dimensional representation is irreducible. Now let V have dimension n and suppose that every representation of dimension $< n$ is isomorphic to a direct sum of irreducible representations. If V is irreducible we are done. Otherwise, we let $\{0\} \neq U \subsetneq V$ be a proper subrepresentation. Maschke's theorem implies that $V \cong U \oplus W$ for some subrepresentation W of V, and both U and W have dimension strictly less than n. By the inductive hypothesis, U and W are isomorphic to direct sums of irreducible representations. Therefore V is also isomorphic to a direct sum of irreducible representations.

Exercise 4.
(1) Let G be a group and
$$\chi : G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$$
a group homomorphism (i.e. a one-dimensional matrix representation). Show that if $g, h \in G$ then $\chi(g) = \chi(hgh^{-1})$.
(2) We let $G = S_n$. Let $2 \leq j \leq n$ be an integer. Show that there is an element $h \in S_n$ such that $h(12)h^{-1} = (1j)$. Show moreover that if $g \in S_n$ is any transposition (i.e. $g = (jk)$ for $j \neq k$) then there exists an $h \in G$ such that $h(12)h^{-1} = g$.
(3) Show that there are only two one-dimensional representations of S_n (up to isomorphism), given by the trivial map $S_n \to \{1\}$ and the sign homomorphism $S_n \to \{\pm 1\}$. *Recall that every element of S_n is a product of transpositions.*

Solution 4.
(1) We have $\chi(hgh^{-1}) = \chi(h)\chi(g)\chi(h^{-1})$. Since χ is a homomorphism, we have $\chi(h)^{-1}$, and we get $\chi(hgh^{-1}) = \chi(h)\chi(g)\chi(h)^{-1}$. Since $\mathbb{C}^\times$ is commutative, we can rearrange this to get $\chi(hgh^{-1}) = \chi(g)$.
(2) If $j = 2$ we can just let $h = e$. So suppose $3 \leq j \leq n$. Then we set $h = (2j)$. Now $h(12)h^{-1} = (2j)(12)(2j)$ which is a permutation which swaps 1 and j and fixes everything else. So we have $h(12)h^{-1} = (1j)$.
For the 'moreover' statement we just apply the same trick one more time. We can assume that $j \neq 1$ (otherwise we can just swap j and k). So first we find h_1 such that $h_1(12)h_1^{-1} = (1j)$. Now if $k = 1$ we are done. Otherwise, we let $h_2 = (1k)$ and consider $h_2h_1(12)(h_2h_1)^{-1} = (1k)(1j)(1k) = (jk)$.
(3) Up to isomorphism, a one-dimensional representation V is determined by the homomorphism $\chi : S_n \to \mathbb{C}^\times$ satsifying $\rho_V(g)v = \chi(g)v$ for $g \in G$ and $v \in V$. So we need to show that the trivial map and the sign homomorphism are the only two homomorphisms $\chi : S_n \to \mathbb{C}^\times$. Fix such a homomorphism χ. From part (2), we know that every transposition is conjugate to (12). Applying part (1) we deduce that $\chi(jk) = \chi(12)$ for every $j \neq k$. Since (12) has order 2 we have $\chi(12) = 1$ or -1. If $\chi(12) = 1$ then $\chi(jk) = 1$ for all $j \neq k$. Since every element of S_n is a product of transpositions, we have $\chi(g) = 1$ for all $g \in S_n$. So χ is trivial. Alternatively, we have $\chi(12) = -1$, so $\chi(jk) = -1$ for all $j \neq k$, and χ is the sign homomorphism.

Exercise 5.
(1) Let G be a finite group. Write $Z(G)$ for the centre of the group:
$$Z(G) = \{z \in G : zg = gz \forall g \in G\}.$$
Note that $Z(G)$ is a subgroup of G. Let V be an irreducible representation of G. Show that for each $z \in Z(G)$ there exists $\lambda_z \in \mathbb{C}$ such that
$$\rho_V(z)v = \lambda_z v$$

for all $v \in V$.

(2) Suppose V is a faithful irreducible representation of G. Show that $Z(G)$ is a cyclic group. *Hint: A finite subgroup of* $\mathbb{C}^\times$ *is cyclic.*

Solution 5. (1) If $z \in Z(G)$ then $\rho_V(z)$ is a G-linear map from V to V. Schur's lemma implies that $\rho_V(z)$ is multiplication by a scalar λ_z (since V is irreducible).

(2) The map $z \mapsto \lambda_z$ gives a homomorphism $Z(G) \to \mathbb{C}^\times$. Since V is faithful we know that $\lambda_z = 1$ if and only if $z = e$. So this is an injective homomorphism. Therefore $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$ (namely, the image of the homomorphism $z \mapsto \lambda_z$). Now the hint implies that $Z(G)$ is cyclic.

By the way, here is a quick proof of the hint: let H be a finite subgroup of $\mathbb{C}^\times$. Then for every $h \in H$ we have $h^{|H|} = 1$. So the elements of H give $|H|$ roots of the degree $|H|$ polynomial $z^{|H|} - 1$. But this polynomial has exactly $|H|$ roots, the complex numbers $e^{2\pi i j/|H|}$. This means that H is the cyclic group of $|H|$th roots of unity.

Exercise 6. (1) Let D_{2n} be the dihedral group of order $2n$, generated by a rotation s of order n and a reflection t of order 2. Recall that we have $tst = s^{-1}$. Let $\zeta \in \mathbb{C}$ be an nth root of unity and let V_ζ be the representation of D_{2n} on the vector space $\mathbb{C}^2$ (with the standard basis) given by

$$\rho_{V_\zeta}(s) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad \rho_{V_\zeta}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Verify that this defines a representation of D_{2n} on V.

Show that if $\zeta \neq \pm 1$ then this representation is irreducible.

(2) What are the one-dimensional matrix representations of D_{2n}?

(3) Show that if n is even, there are $(n+6)/2$ isomorphism classes of irreducible representations of D_{2n}: 4 of dimension one and $(n-2)/2$ of dimension 2.

(4) Show that if n is odd, there are $(n+3)/2$ isomorphism classes of irreducible representations of D_{2n}: 2 of dimension one and $(n-1)/2$ of dimension 2.

Solution 6. (1) To verify that we have defined a representation, we need to check the relation $TST = S^{-1}$ holds for the matrices

$$S = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$S^{-1} = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}$$

and also

$$TST = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}$$

so we have indeed defined a representation.

Suppose $\zeta \neq \pm 1$, so $\zeta \neq \zeta^{-1}$ and the matrix S has distinct eigenvalues. If V_ζ is reducible, then the matrices S, T have a simultaneous non-zero eigenvector. But the eigenspaces for S are $\mathbb{C} \cdot (0,1)$ and $\mathbb{C} \cdot (1,0)$, and these do not contain any non-zero eigenvectors for T. So there is no simultaneous non-zero eigenvector and V_ζ is irreducible.

(2) A one dimensional matrix representation is a homomorphism

$$\chi : D_{2n} \to \mathbb{C}^\times.$$

Applying Question 4 part (1) we see that $\chi(s) = \chi(s)^{-1}$. So we have $\chi(s) = 1$ or -1 and $\chi(t) = 1$ or -1. If n is odd then $\chi(s)^n = 1$, and this forces $\chi(s) = 1$. So we have two cases:

- If n is odd, then there are two one dimensional matrix reps, given by the trivial map $\chi : D_{2n} \to \{1\}$ and the homomorphism $\chi : D_{2n} \to \{\pm 1\}$ defined by $\chi(s) = 1, \chi(t) = -1$.
- If n is even, there are four one dimensional matrix reps, given by the homomorphisms χ defined by $\chi(s) = \pm 1$, $\chi(t) = \pm 1$, where we can take any of the four combinations of signs.

(3) Suppose n is even. Then we have written down four one dimensional representations of D_{2n}. These four representations are irreducible (since they are one dimensional) and non-isomorphic (since they have distinct matrix representations, and equivalent one dimensional matrix reps are equal). As ζ runs over the set $\Sigma = \{e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{2(n/2-1)\pi i/n}\}$ we get $n/2 - 1 = (n-2)/2$ irreducible two dimensional representations V_ζ. They are all non-isomorphic, because the eigenvalues of $\rho_{V_\zeta}(s)$ are $\{\zeta, \zeta^{-1}\}$ and these sets of eigenvalues are all distinct as ζ ranges over Σ (if two representations are isomorphic, the action of $g \in G$ necessarily has the same eigenvalues on both representations).

So we have written down 4 isomorphism classes of one dimensional reps and $(n-2)/2$ isomorphism classes of two dimensional irreps. Recall that we have an equality $2n = \sum_{i=1}^r d_i^2$ where the d_i are the dimensions of the irreps of D_{2n}. Since $4 + (n-2)/2 \times 4 = 2n$ we have found all the isomorphism classes of irreps.

(4) The case with n odd is very similar. Now there are only two one dimensional reps of D_{2n}. As ζ runs over the set $\Sigma = \{e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{(n-1)\pi i/n}\}$ we get $(n-1)/2$ non-isomorphic two dimensional irreps V_ζ. Adding up the squares of the dimensions, we have $2 + (n-1)/2 \times 4 = 2n$, so we have found all the isomorphism classes.