# M3/4/5P12 PROBLEM SHEET 1

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**Exercise 1.** (1) Let $G = C_4 \times C_2 = \langle s, t : s^4 = t^2 = e, st = ts \rangle$. Let $V = \mathbb{C}^2$ with the standard basis. Consider the linear transformations of $V$ defined by the matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Verify that sending $s$ to $S$ and $t$ to $T$ defines a representation of $G$ on $V$. Is this representation faithful?

(2) Now let

$$Q = \begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} -1 & 0 \\ i+1 & 1 \end{pmatrix}.$$

Verify that sending $s$ to $Q$ and $t$ to $R$ also defines a representation of $G$ on $V$. Is this representation faithful?

(3) Show that $S$ is conjugate to $Q$ and $T$ is conjugate to $R$. Are the two representations we have defined isomorphic?

**Solution 1.** (1) To show that we have defined a representation of $G$ we need to check that the matrices $S$ and $T$ satisfy the relations which are given for $s$ and $t$. So we need to check that

$$S^4 = T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$ST = TS.$$

These should be straightforward computations.

The representation *is* faithful. We need to check that the linear map associated to an element $g \in G$ is the identity if and only if $g$ is the identity. We can write $g = s^a t^b$, so $\rho_V(g)$ has matrix

$$S^a T^b = \begin{pmatrix} (-1)^b & 0 \\ 0 & i^4 \end{pmatrix}.$$

This matrix is the identity matrix if and only if $a$ is divisible by 4 and $b$ is divisible by 2. But if these divisibilities hold then $s^a = t^b = e$ and so $g = e$.

(2) Again we just need to check the relations. We have

$$Q^2 = \begin{pmatrix} -1 & 0 \\ i+1 & 1 \end{pmatrix} = R$$

and

$$Q^4 = R^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We also have $QR = Q^3 = RQ$.

Finally, this representation is not faithful. We have

$$Q^2 R = R^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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so $s^2t$ is send to the identity under this representation. Since $s^2t \neq e$ the representation is not faithful.

(3) To show that $S$ is conjugate to $Q$ and $T$ is conjugate to $R$ you can either find explicit matrices conjugating one to the other, or compute eigenvalues. For example $S$ and $Q$ both have eigenvalues $1, i$, which are distinct, so both $S$ and $Q$ are conjugate to the diagonal matrix $\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$ and are therefore conjugate to each other. The two representations are not isomorphic because one is faithful and the other is not. Call the first representation $\rho$ and the second $\sigma$. The 1-eigenspace for $Q$ is equal to the 1-eigenspace for $R$. An isomorphism of representations from $\sigma$ to $\rho$ would take this common 1-eigenspace for $Q = \sigma(s)$ and $R = \sigma(t)$ to a common 1-eigenspace for $S = \rho(s)$ and $T = \rho(t)$, but there is no non-zero simultaneous eigenvector for $S$ and $T$ with eigenvalue 1.

**Exercise 2.** (1) Let $G$ be a finite group, and $(V, \rho_V)$ a representation of $G$, with $V$ a finite dimensional complex vector space. Let $g$ be an element of $G$. Show that there is a positive integer $n \geq 1$ such that $\rho_V(g)^n = \mathrm{id}_V$. What can you conclude about the minimal polynomial of $\rho_V(g)$?

(2) Show that $\rho_V(g)$ is diagonalisable.

**Solution 2.** (1) Since $G$ is a finite group there is a positive integer $n \geq 1$ such that $g^n = e$. This implies that $\rho_V(g)^n = \rho_V(g^n) = \rho_V(e) = \mathrm{id}_V$. Since the linear map $\rho_V(g)$ satisfies the polynomial $X^n - 1$, we conclude that the minimal polynomial of $\rho_V(g)$ divides $X^n - 1$.

(2) Putting $\rho_V(g)$ in Jordan normal form, we get that the minimal polynomial of $\rho_V(g)$ is equal to $\prod_{i=1}^{d}(X - \lambda_i)^{e_i}$ where $\lambda_1, \ldots \lambda_d$ are the eigenvalues of $\rho_V(g)$ and $e_i$ is the size of the largest Jordan block with diagonal entry $\lambda_i$. Now $X^n - 1 = \prod_{j=1}^{n}(X - \zeta^n)$ where $\zeta = e^{2\pi i/n}$ is a primitive $n$th root of unity. Since the minimal polynomial of $\rho_V(g)$ divides $X^n - 1$ it is a product of distinct linear factors. So all the numbers $e_i$ are equal to 1 and $\rho_V(g)$ is diagonalisable.

If you don't like using Jordan normal form you can also prove directly that if the minimal polynomial of a linear map $f : V \to V$ is a product of distinct factors $\prod_{i=1}^{d}(X - \lambda_i)$ then $f$ is diagonalisable. First show that the polynomial

$$g(X) = \sum_{j=1}^{d} \prod_{i \neq j, i=1}^{d} \frac{(X - \lambda_i)}{\lambda_j - \lambda_i} = 1$$

*Hint: for each $j = 1, \ldots, d$ we have $g(\lambda_j) = 1$ but $g$ has degree $d - 1$.* Then show that the linear map

$$\prod_{i \neq j, i=1}^{d} \frac{(f - \lambda_i)}{\lambda_j - \lambda_i}$$

has image $V_j \subset V$ such that $f|_{V_j}$ is multiplication by $\lambda_j$. The fact that $g(X) = 1$ implies that $V_1, \ldots V_d$ span $V$ and so $V$ has a basis of eigenvectors for $f$.

**Exercise 3.** (1) Consider $S_3$ acting on $\Omega = \{1, 2, 3\}$ and write $V$ for the associated permutation representation $\mathbb{C}\Omega$. Write down the matrices giving the action of $(123), (23)$ with respect to the standard basis $([1], [2], [3])$ of $V$.

(2) Write $U$ for the subspace of $V$ consisting of vectors $\{\lambda_1[1] + \lambda_2[2] + \lambda_3[3] : \lambda_1 + \lambda_2 + \lambda_3 = 0\}$. Show that $U$ is mapped to itself by the action of $S_3$. Find a basis of $U$ with respect to which the action of (23) is given by a diagonal matrix and write down the matrix giving the action of (123) with respect to this basis.

Can you find a basis of $U$ with respect to which the actions of both (23) and (123) are given by diagonal matrices?

**Solution 3.** (1) The action of (123) is given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The action of (23) is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(2) Let $g \in S_3$ and suppose $\lambda_1 + \lambda_2 + \lambda_3 = 0$. We have

$$g \cdot (\lambda_1[1] + \lambda_2[2] + \lambda_3[3]) = \lambda_1[g \cdot 1] + \lambda_2[g \cdot 2] + \lambda_3[g \cdot 3] = \lambda_{g^{-1}.1}[1] + \lambda_{g^{-1}.2}[2] + \lambda_{g^{-1}.3}[3]$$

and $\lambda_{g^{-1}.1} + \lambda_{g^{-1}.2} + \lambda_{g^{-1}.3} = \lambda_1 + \lambda_2 + \lambda_3$. So if $v = \lambda_1[1] + \lambda_2[2] + \lambda_3[3]$ is in $U$ then $g \cdot v$ is in $U$. Let $v_1 = 2[1] - [2] - [3]$ and $v_2 = -[2] + [3]$. Then $v_1, v_2$ give a basis for $U$ and we have $(23) \cdot v_1 = v_1$, $(23) \cdot v_2 = -v_2$. So this gives a basis of $U$ with respect to which the action of (23) is diagonal. If we modify the basis by multiplying $v_1$ or $v_2$ by a non-zero scalar we get a new basis, but these are the only possible choices.

Let's compute the action of (123) with respect to the basis $v_1, v_2$. We have

$$(123)v_1 = 2[2] - [3] - [1] = -1/2v_1 - 3/2v_2$$

and

$$(123)v_2 = -[3] + [1] = 1/2v_1 - 1/2v_2.$$

So we get a matrix

$$\begin{pmatrix} -1/2 & 1/2 \\ -3/2 & -1/2 \end{pmatrix}.$$

Note that if we rescale our basis to $v_1, \sqrt{3}v_2$ we get a matrix

$$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

which is the same as the matrix we wrote down when giving the two-dimensional rep of $S_3$ arising from viewing $S_3$ as the symmetry group of the triangle.

Finally, the only bases of $U$ with respect to which the action of (23) is diagonal are given by $\alpha v_1, \beta v_2$, and the action of (123) with respect to these bases is not diagonal. So it is not possible to find a basis of $U$ with respect to which the actions of both (23) and (123) are diagonal.

**Exercise 4.** (1) Let $V, W$ be two representations of $G$ and $f : V \to W$ an invertible $G$-linear map. Show that $f^{-1}$ is $G$-linear.

(2) Show that a composition of two $G$-linear maps is $G$-linear.

(3) Deduce that 'being isomorphic' is an equivalence relation on representations of a group $G$.

**Solution 4.** (1) We want to show that
$$f^{-1} \circ \rho_W(g) = \rho_V(g) \circ f^{-1}$$
for all $g \in G$. Equivalently, we want to show that
$$\rho_V(g^{-1}) \circ f^{-1} \circ \rho_W(g) = f^{-1}$$
for all $g \in G$. Composing with $f$ on both sides, we see that it's enough to show that
$$f \circ \rho_V(g^{-1}) \circ f^{-1} \circ \rho_W(g) = \mathrm{id}_W$$
and since $f$ is $G$-linear we can simplify the left hand side
$$f \circ \rho_V(g^{-1}) \circ f^{-1} \circ \rho_W(g) = \rho_W(g^{-1}) \circ f \circ f^{-1} \circ \rho_W(g) = \rho_W(g^{-1}) \circ \rho_W(g) = \mathrm{id}_W\,.$$

(2) Suppose $f_1 : U \to V$ and $f_2 : V \to W$ are $G$-linear maps. We want to show that the composition $f_2 \circ f_1 : U \to W$ is $G$-linear. We have
$$f_2 \circ f_1 \circ \rho_U(g) = f_2 \circ \rho_V(g) \circ f_1 = \rho_W(g) \circ f_2 \circ f_1$$
where we use $G$-linearity of $f_1$ for the first inequality and $G$-linearity of $f_2$ for the second. We conclude that $f_2 \circ f_1$ is $G$-linear.

(3) Being isomorphic is reflexive, since the identity map is a $G$-linear isomorphism. Part (1) shows that being isomorpic is symmetric, since if $f : V \to W$ is a $G$-linear isomorphism, the inverse $f^{-1}$ is a $G$-linear isomorphism from $W$ to $V$. Finally, transitivity follows from part (2).

**Exercise 5.** (1) Let $G, H$ be two finite groups, and let $f : G \to H$ be a group homomorphism. Suppose we have a representation $V$ of $H$. Show that $\rho_V \circ f : G \to \mathrm{GL}(V)$ defines a representation of $G$. We call this representation the *restriction* of $V$ from $H$ to $G$ along $f$, written $\mathrm{Res}_f(V)$.

(2) Let $S_n$ act on the set of cosets $\Omega = \{eA_n, (12)A_n\}$ for the alternating group $A_n \subset S_n$ by left multiplication. We get a two-dimensional representation $\mathbb{C}\Omega$ of $S_n$. Show that $\mathbb{C}\Omega$ is isomorphic to $\mathrm{Res}_{sgn}(V)$ where $sgn : S_n \to \{\pm 1\}$ is the sign homomorphism[1] and $V$ is the regular representation of $\{\pm 1\}$.

**Solution 5.** (1) We just need to check that $\rho_V \circ f$ is a group homomorphism. Since $f$ and $\rho_V$ are group homomorphisms, the composition $\rho_V \circ f$ is too.

(2) We need to write down an $S_n$-linear isomorphism
$$\alpha : \mathbb{C}\Omega \to \mathrm{Res}_{sgn}(V).$$
We define $\alpha([eA_n]) = [+1]$ and $\alpha([(12)A_n]) = [-1]$. Then $\alpha$ is clearly an invertible linear map. It remains to check that $\alpha$ is $S_n$-linear. For $g \in S_n$ we have $gA_n = eA_n$ if $sgn(g) = +1$ and $gA_n = (12)A_n$ if $sgn(g) = -1$. So we can check that $\alpha(g \cdot [eA_n]) = \alpha([gA_n]) = (sgn(g))\alpha([eA_n])$ and $\alpha(g \cdot [(12)A_n]) = \alpha([g(12)A_n]) = (sgn(g))\alpha([(12)A_n])$. This shows that $\alpha$ is $S_n$-linear, as desired.

**Exercise 6.** (1) Let $C_n = \langle g : g^n = e \rangle$ be a cyclic group of order $n$. Let $V_{reg}$ be the regular representation of $C_n$. What is the matrix for the action of $g$ on $V_{reg}$, with respect to the basis $[e], [g], \ldots [g^{n-1}]$? What are the eigenvalues of this matrix?

(2) Find a basis for $V_{reg}$ consisting of eigenvectors for $\rho_{V_{reg}}(g)$.

[1]taking even permutations to $+1$ and odd permutations to $-1$

(3) Let $G$ be a finite Abelian group, and let $V$ be a representation of $G$. Show that $V$ has a basis consisting of simultaneous eigenvectors for the linear maps $\{\rho_V(g) : g \in G\}$. *Hint: recall the fact from linear algebra that a commuting family of diagonalisable linear operators is simultaneously diagonalisable.*

**Solution 6.** (1) The matrix of $g$ has 1 in the entries below the diagonal, and a 1 in the top right hand corner. Let's compute the eigenvalues. Suppose $v$ is a non-zero eigenvector for $g$ with eigenvalue $\mu$. We write $v = \sum_{i=0}^{n-1} \lambda_i [g^i]$ we have

$$g \cdot \left( \sum_{i=0}^{n-1} \lambda_i [g^i] \right) = \mu \sum_{i=0}^{n-1} \lambda_i [g^i].$$

The left hand side is equal to

$$\sum_{i=0}^{n-1} \lambda_i [g^{i+1}]$$

so equating coefficients we get $\mu\lambda_1 = \lambda_0$, $\mu\lambda_2 = \lambda_1, \dots, \mu\lambda_{n-1} = \lambda_{n-2}$ and $\mu\lambda_0 = \lambda_{n-1}$. Putting everything together we find that $\lambda_i = \mu^{n-i}\lambda_0$ for $i = 0\lambda n - 1$. In particular we have $\mu^n = 1$ and given an $n$th root of unity $\mu$ $\sum_{i=0}^{n-1} \mu^{n-i}[g^i]$ is a non-zero eigenvector for $g$ with eigenvalue $\mu$. So there are $n$ distinct eigenvalues, each of the $n$th roots of unity.

(2) I already did this in part (1): the basis is given by the $n$ vectors $v_\mu = \sum_{i=0}^{n-1} \mu^{n-i}[g^i]$ for $\mu$ an $n$th root of unity.

(3) The linear operators $\rho_V(g)$ are all diagonalisable, by Exercise 2, and they all commute with each other because $G$ is Abelian, so for $g, h \in G$ $gh = hg$ and therefore $\rho_V(g)\rho_V(h) = \rho_V(h)\rho_V(g)$. So the linear algebra fact quoted in the exercise tells us that we have a basis of simultaneous eigenvectors.