International Workshop On Combinatorial Algorithms
IWOCA

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Non-preemptive tree packing

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presented at IWOCA 2021
06 July 2021

The *tree packing problem* of Nash-Williams is a classical problem [2]. Here, one is given a graph with nonnegative integral edge weights $w : E \to \mathbb{N}_0$. The goal is to pack a maximal number of spanning trees into the graph, such that for every edge, the number of spanning trees using e does not exceed $w(e)$. The problem can be solved in strongly polynomial time [3].

The above problem can be interpreted as a scheduling problem: Every edge is a resource that can be scheduled for a total of $w(e)$ time units, where preemption of edges is allowed. (See Figure 1 as an illustration, here edge e_3 gets preempted at time 1 and resumed at time 2.) The goal is that the scheduled edges connect the graph for a maximal amount of time.

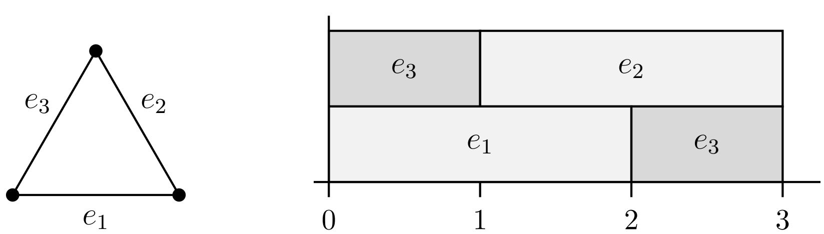

Figure 1: The three edges in the graph on the left hand side have weights $w(e_1) = w(e_2) = w(e_3) = 2$. The schedule on the right hand side keeps the graph connected for a total of three time units.

The non-preemptive version of tree packing. Lendl, Woeginger and Wulf introduced a non-preemptive variant of the above tree packing problem, where the execution of edges must not be preempted [1]: Every edge e is activated at some time point $\tau(e)$ chosen by the scheduler, and then remains active without interruption during the full time interval $[\tau(e),\, \tau(e) + w(e)]$. The objective is again to activate the edges in such a way that the graph remains connected for the longest possible overall time. The resulting combinatorial optimization problem is called *non-preemptive* *tree packing* (N-TreePack for short), and the optimal objective value for a graph $G = (V, E)$ with edge weights $w : E \to \mathbb{N}_0$ will be denoted $\mathsf{ntp}(G, w)$.

In the example in Figure 1, one sees that one can pack three spanning trees, but $\mathsf{ntp}(G, w) = 2$.

Open Problems

1. **Is there a constant-factor approximation for** N-TreePack**?**
   - It is known that there exists a $\Theta(|V|)$-approximation, and that there exists no 7/6-approximation (unless P=NP) [1].
2. **Is the integrality gap unbounded?**
   - The *integrality gap* of an instance is the quotient of the solution of the spanning tree packing problem, and the non-preemptive spanning tree packing problem. (So far, we don't even have examples where the gap is larger than 2!)
3. **What is the complexity to decide whether** $\mathsf{ntp}(G, w) \geq \beta$ **for** $\beta \in \{4, 5, 6\}$**?**
   - It was shown that deciding whether $\mathsf{ntp}(G, w) \geq 7$ is NP-complete, and deciding whether $\mathsf{ntp}(G, w) \geq 3$ can be done in polynomial time [1].
4. **Can** N-TreePack **be solved in polynomial time if** $w(e) \in \{1, 2\}$ **for every edge?**
   - It can not be solved in polynomial time, if $w(e) \in \{1, \ldots, 6\}$ for every edge [1].

References

[1] S. Lendl, G. J. Woeginger and L. Wulf: Non-preemptive Tree Packing. *Flocchini P., Moura L. (eds) Combinatorial Algorithms. IWOCA 2021.*, LNCS 12757:456–468, 2021.

[2] C. S. J. Nash-Williams: Edge-disjoint spanning trees of finite graphs. *Journal of the London Mathematical Society*, 1(1): 445–450, 1961.

[3] W. H. Cunningham: Optimal attack and reinforcement of a network. *Journal of the ACM 32*, 549–561, 1985.