International Workshop On
Combinatorial Algorithms
IWOCA

Decidability of set decipherability over directed figures

Włodzimierz Moczurad
Faculty of Mathematics and Computer Science, Jagiellonian University
wkm@ii.uj.edu.pl

21 July 2012
updated on 25 March 2013

1 The open problem

The open problem is stated here; definitions are given in the next section. Let Σ be a finite, non-empty alphabet and let $m : \Sigma \times \Sigma \to \Sigma$ be an associative merging function.

- Problem 1: Is the following question decidable: Given a two-sided code X over Σ, is X an SD m-code?
- Problem 2, a possible restricted variant of the above: Is the following question decidable: Given a two-sided code X over Σ with parallel translation vectors, is X an SD m-code?

2 Definitions

Let Σ be a finite, non-empty alphabet. A *translation* by vector $u \in \mathbb{Z}^2$ is denoted by tr_u.

Definition 1 (Directed figure) *Let $D \subseteq \mathbb{Z}^2$ be finite and non-empty, $b, e \in \mathbb{Z}^2$ and $l : D \to \Sigma$. A quadruple $f = (D, b, e, l)$ is a* directed figure *(over Σ) with*

$$\begin{array}{lrcl} \text{domain} & \mathrm{dom}(f) & = & D, \\ \text{start point} & \mathrm{begin}(f) & = & b, \\ \text{end point} & \mathrm{end}(f) & = & e, \\ \text{labelling function} & \mathrm{label}(f) & = & l. \end{array}$$

Translation vector *of f is defined as* $\mathrm{tran}(f) = \mathrm{end}(f) - \mathrm{begin}(f)$. *Additionally, the* empty directed figure *ε is defined as $(\emptyset, (0,0), (0,0), \{\})$, where $\{\}$ denotes a function with an empty domain.*

The set of all directed figures over Σ is denoted by $\Sigma^{\diamond}$. Two directed figures x, y are *equal* (denoted by $x = y$) if there exists $u \in \mathbb{Z}^2$ such that $y = (\mathrm{tr}_u(\mathrm{dom}(x)), \mathrm{tr}_u(\mathrm{begin}(x)), \mathrm{tr}_u(\mathrm{end}(x)), \mathrm{tr}_u(\mathrm{label}(x)))$. Thus, we actually consider figures up to translation.

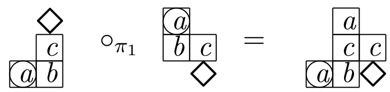
Definition 2 **(m-catenation)** *Let $x = (D_x, b_x, e_x, l_x)$ and $y = (D_y, b_y, e_y, l_y)$ be directed figures. An m-*catenation *of x and y with respect to a* merging function $m : \Sigma \times \Sigma \to \Sigma$ *is defined as*

$$x \circ_m y = (D_x \cup \mathrm{tr}_{e_x - b_y}(D_y), b_x, \mathrm{tr}_{e_x - b_y}(e_y), l),$$

where

$$l(z) = \begin{cases} l_x(z) & \text{for } z \in D_x \setminus \mathrm{tr}_{e_x-b_y}(D_y), \\ \mathrm{tr}_{e_x-b_y}(l_y)(z) & \text{for } z \in \mathrm{tr}_{e_x-b_y}(D_y) \setminus D_x, \\ m(l_x(z), \mathrm{tr}_{e_x-b_y}(l_y)(z)) & \text{for } z \in D_x \cap \mathrm{tr}_{e_x-b_y}(D_y). \end{cases}$$

Example 1 *Let π_1 be the projection onto the first argument.*



Observe that $\circ_m$ is associative if and only if m is associative.

3 Codes

Note that by a *code* (over Σ, with no further attributes) we mean any finite non-empty subset of $\Sigma^\diamond \setminus \{\varepsilon\}$.

Definition 3 (Uniquely decipherable (UD) m-code) *Let X be a code over Σ. X is a* uniquely decipherable m-code*, if for any $x_1, \dots, x_k$, $y_1, \dots, y_l \in X$ the equality $x_1 \circ_m \cdots \circ_m x_k = y_1 \circ_m \cdots \circ_m y_l$ implies that $(x_1, \dots, x_k)$ and $(y_1, \dots, y_l)$ are equal as sequences.*

Definition 4 (Multiset decipherable (MSD) m-code) *Let X be a code over Σ. X is a* multiset decipherable m-code*, if for any $x_1, \dots, x_k$, $y_1, \dots, y_l \in X$ the equality $x_1 \circ_m \cdots \circ_m x_k = y_1 \circ_m \cdots \circ_m y_l$ implies that $\{\!\{x_1, \dots, x_k\}\!\}$ and $\{\!\{y_1, \dots, y_l\}\!\}$ are equal as multisets.*

Definition 5 (Set decipherable (SD) m-code) *Let X be a code over Σ. X is a* set decipherable m-code*, if for any $x_1, \dots, x_k$, $y_1, \dots, y_l \in X$ the equality $x_1 \circ_m \cdots \circ_m x_k = y_1 \circ_m \cdots \circ_m y_l$ implies that $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_l\}$ are equal as sets.*

Definition 6 (Numerically decipherable (ND) m-code) *Let X be a code over Σ. X is a* numerically decipherable m-code*, if for any $x_1, \dots, x_k$, $y_1, \dots, y_l \in X$ the equality $x_1 \circ_m \cdots \circ_m x_k = y_1 \circ_m \cdots \circ_m y_l$ implies $k = l$.*

Definition 7 (Two-sided and one-sided codes) *Let $X = \{x_1, \dots, x_n\}$ be a code over Σ. If there exist $\alpha_1, \dots, \alpha_n \in \mathbb{N}$, not all equal to zero, such that $\sum_{i=1}^{n} \alpha_i \mathrm{tran}(x_i) = (0, 0)$, then X is called* two-sided*. Otherwise it is called* one-sided.

This condition can be intrepreted geometrically as follows: Translation vectors of a two-sided code do not fit in an open half-plane. For a one-sided code, there exists a line passing through $(0, 0)$ such that all translation vectors are on one side of it.

4 Summary of decidability results

The problem we consider is the decidability of decipherability testing, *i.e.* given a code X with a specific geometry (one-sided, two-sided, two-sided with parallel translation vectors), is X an m-code of a specific kind (UD, MSD, ND, SD)? The table below summarizes the decidability status for different combinations of code geometry and kind. Decidable cases are marked with a + and combinations that are still open are denoted with a question mark.

		UD	MSD	ND	SD
1	One-sided m-codes	+	+	+	+
2	Two-sided m-codes	+	+	+	?
3	Two-sided m-codes with parallel vectors	+	+	+	?

5 Remarks

Note that the positive decidability cases depicted in lines 2 and 3 are trivial: two-sided UD, MSD or ND m-codes do not exist. For other decidable combinations, respective proofs lead to effective verification algorithms. See [4, 3, 5, 1].

On the other hand, the case of two-sided SD m-codes is non-trivial; both SD and not-SD codes of this kind exist. However, none of the proof techniques we have used so far can be adapted to this case.

Another type of catenation, without merging function, can also be considered and it leads to different decidability results: all decipherability kinds are decidable for one-sided codes and undecidable for two-sided codes. See [2]

References

[1] Kolarz, M.: The code problem for directed figures. Theoretical Informatics and Applications RAIRO 44(4), 489–506 (2010)

[2] Kolarz, M.: Directed figure codes: Decidability frontier. In: COCOON 2010. Lecture Notes in Computer Science, vol. 6196, pp. 530–539. Springer (2010)

[3] Kolarz, M., Moczurad, W.: Directed figure codes are decidable. Discrete Mathematics and Theoretical Computer Science 11(2), 1–14 (2009)

[4] Kolarz, M., Moczurad, W.: Multiset, set and numerically decipherable codes over directed figures. In: IWOCA 2012. Lecture Notes in Computer Science, vol. 7643, pp. 224–235. Springer (2012)

[5] Moczurad, W.: Directed figure codes with weak equality. In: Intelligent Data Engineering and Automated Learning IDEAL 2010. Lecture Notes in Computer Science, vol. 6283, pp. 242–250. Springer (2010)