Graphs with no equal length cycles

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Let $f(n)$ be the maximum number of edges in a graph on $n$ vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see J.A. Bondy and U.S.R. Murty [1], p.247, Problem 11). Y. Shi [9] proved that

$$f(n) \geq n + [(\sqrt{8n - 23} + 1)/2]$$

for $n \geq 3$. E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] proved that

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

Chunhui Lai [7] proved that

$$\liminf_{n \to \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2.4}.$$ 

and conjectured that

$$\lim_{n \to \infty} \frac{f(n) - n}{\sqrt{n}} = \sqrt{2.4}.$$ 

We think it is difficult to prove this conjecture. It would be nice to prove the following conjecture:

**Conjecture [6].**

$$\liminf_{n \to \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$ 

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on $n$ vertices in which no two cycles have the same length.

In 1988, Shi[9] proved that

For every integer $n \geq 3$, $f_2(n) \leq n + \lfloor \sqrt{8n - 15} - 3 \rfloor$.

In 1998, Guantao Chen, Jeno Lehel, Michael S.Jacobson, and Warren E. Shreve [3] proved that $f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$.
In 2001, E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] improved this lower bound significantly:

\[ f_2(n) \geq n + \sqrt{n} - O\left(\frac{n^{9/20}}{n}\right). \]

and conjectured that

\[ \lim_{n \to \infty} \frac{f_2(n)}{\sqrt{n}} = 1. \]

It is easy to see that this Conjecture implies the (difficult) upper bound in the Erdős-Turan Theorem [4][5](see [2]).

Markström [8] raised the problem of determining the maximum number of edges in a hamiltonian graph on \( n \) vertices with no repeated cycle lengths.

Let \( S_n \) be the set of simple graphs on \( n \) vertices in which no two cycles have the same length. A graph \( G \) in \( S_n \) is called a simple maximum cycle-distributed graph (simple MCD graph) if there exists no graph \( G' \) in \( S_n \) with \( |E(G')| > |E(G)| \). A planar graph \( G \) is called a generalized polygon path (GPP) if \( G^* \) formed by the following method is a path: corresponding to each interior face \( f \) of \( \tilde{G} \) (\( \tilde{G} \) is a plane graph of \( G \)) there is a vertex \( f^* \) of \( G^* \); two vertices \( f^* \) and \( g^* \) are adjacent in \( G^* \) if and only if the intersection of the boundaries of the corresponding interior faces of \( \tilde{G} \) is a simple path of \( \tilde{G} \). Shi, Yong-Bing; Tang, Yin-Cai; Tang, Hua; Gong, Ling-Liu; Xu, Li[10] prove that there exists a simple MCD graph on \( n \) vertices such that it is a 2-connected graph being not a GPP if and only if \( n \in \{10, 11, 14, 15, 16, 21, 22\} \). Shi, Yong-Bing; Tang, Yin-Cai; Tang, Hua; Gong, Ling-Liu; Xu, Li[10] also prove that, by discussing all the natural numbers except for 75 natural numbers, there are exactly 18 natural numbers, for each \( n \) of which, there exists a simple MCD graph on \( n \) vertices such that it is a 2-connected graph.(see[10])

References


