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On the Minimum Number of Plus Operators in Expressions of Series-Parallel Graphs and in Read-Once Functions

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We consider a *labeled two-terminal directed acyclic graph* in which each edge has a unique label. Each path between the source and the sink (a *spanning path*) in a graph can be represented by a product of all edge labels of the path. We define the sum of edge label products corresponding to all possible spanning paths of a graph G as the *canonical expression* of G. An algebraic expression is called a *graph expression* if it is algebraically equivalent to the canonical expression of a graph. A graph expression consists of edge labels, the operators $+$ (disjoint union) and $\cdot$ (concatenation, also denoted by juxtaposition), and parentheses.

We define the *complexity of an algebraic expression* in two ways. The complexity of an algebraic expression is (i) the total number of labels in the expression including all their appearances (*the first complexity characteristic*) or (ii) the number of plus operators in the expression (*the second complexity characteristic*).

A *series-parallel graph* (Figure 1) is defined recursively so that a single edge is a series-parallel graph and a graph obtained by a *parallel* or a *series composition* of series-parallel graphs is series-parallel.

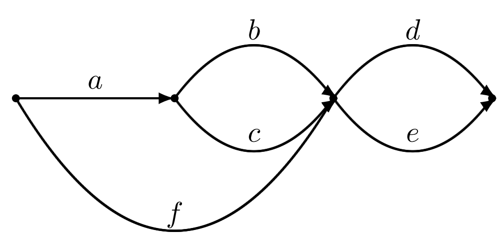

Figure 1: A series-parallel graph.

An expression of a series-parallel graph can be constructed in accordance with its recursive definition. The edges of the graph are represented by their labels while operators $+$ and $\cdot$ indicate parallel and series compositions, respectively. The derived expression is a *linear representation of the series-parallel graph* (*LRSP*). Each label appears in the LRSP only once and, therefore, this representation is the shortest from the perspective of the first complexity characteristic. For example, the LRSP of the graph presented in Figure 1 is $(a(b+c)+f)(d+e)$ whereas its canonical expression is $abd + abe + acd + ace + fe + fd$.

- We conjecture that the LRSP has a minimum number of plus operators and, therefore, it is the shortest representation from the perspective of the second complexity characteristic as well.

This conjecture seems to be proven easily based on the following "evident" claim.

Each irreducible formula that presents an expression of a series-parallel graph is the LRSP of the graph. In other words, every expression of a series-parallel graph can be transformed to the LRSP by factoring using the distributive law in each step.

However, this claim is wrong. The following counterexample due to Endre Boros demonstrates that:

$$\begin{aligned} & (A+B+C+D+E+F)(a+b+c+d+e+f) \\ = \; & (A+B)(a+b)+(B+C)(c+d)+(C+D)(b+f)+(D+E)(a+d)+ \\ & (E+F)(b+c)+(A+F)(d+f)+(B+E)(e+f)+(A+D)(c+e)+(C+F)(a+e). \end{aligned}$$

The left part of this equation is LRSP. Its right part is irreducible and can be converted to the left part only by opening brackets in the first step that leads to increase of the number of plus operators.

The following counterexample due to Alan Woods also disproves the claim above:

$$(a+b)(c+d)(e+f) = a(c(e+f)+df) + b((c+d)f+ce) + (a+b)de.$$

We consider some more problems related to factoring of algebraic expressions of a number of graphs in [1], [2], [3].

A Boolean function is defined as *read-once* if it may be computed by some formula in which no variable occurs more than once (*read-once formula*). The following similar problem is of interest.

- Prove that the number of plus operators in every formula computing a read-once function is greater or equal to the number of plus operators in the read-once formula computing this function.

The problem would be easily solved based on the following "evident" claim.

Each irreducible formula that computes a read-once function is read-once.

However, this claim is disproved by the following counterexample due to Endre Boros (pay attention that in the Boolean algebra $x + x = x$):

$$(a+b+c)(d+e+f) = (a+b)(d+e) + (a+c)(d+f) + (b+c)(e+f).$$

References

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[2] M. Korenblit and V. E. Levit, *Fibonacci Graphs and their Expressions*, arXiv:1305.2647v1, Cornell University Library, 2013, http://arxiv.org/abs/1305.2647v1

[3] M. Korenblit and V. E. Levit, *A One-Vertex Decomposition Algorithm for Generating Algebraic Expressions of Square Rhomboids*, in: Frontiers in Algorithmics and Algorithmic Aspects in Information and Management, Proc. Third Joint Int. Conf., FAW-AAIM 2013, LNCS **7924**, Springer, 2013, 94–105.