International Workshop On
Combinatorial Algorithms
IWOCA

http://www.iwoca.org—Problems section.

Tiling hypercubes with copies of a given graph

Jerrold R. Griggs
Dept. of Mathematics
University of South Carolina
Columbia, SC 29208
USA
griggs@math.sc.edu

presented at IWOCA 2014
updated 25 November, 2014

Co-authors: Kevin Milans, David Offner, and David Stoner

We say that a graph G *tiles* the n-cube graph Q_n, if $V(Q_n)$ can be partitioned into blocks $V_1, V_2, \ldots$ so that, for all i, the induced subgraph on V_i is isomorphic to G. We then propose this graph packing problem: For which graphs G does there exist an n such that G tiles Q_n? When such n exists, we say G is a *paver*.

Easily, when G tiles Q_n, it tiles $Q_{n'}$ for all $n' > n$, so a more precise question is to determine the minimum value of n, denote it $t(G)$, such that G tiles Q_n. In order for G to be a paver, it is necessary that its order $|V(G)|$ is a power of 2, and it has to be *cubical*, which means it is an induced subgraph of some hypercube Q_n. It remains open to give an example of a graph satisfying these necessary conditions that is not a paver.

Restricting attention to graphs G with order a power of 2, we can show that G is a paver if it is a tree, or more generally, a forest (acyclic). Even more, it is a paver if it is unicyclic, where the cycle has even length. Our tilings are linear in the sense that the tiles are translates (viewing $V(Q_n)$ as vectors in F^n, where F is a field of two elements) by vectors t that form a subspace. We are using simple linear algebra and coding theory.

Can one always find a tiling by translates in this way? We now know the answer is no, as we have discovered examples of pavers for which there is no tiling by translates.

We recently found that our theorem for trees was previously discovered by Ramas (1992, JCTB), and a simpler proof of it, more like ours, was given by Mollard (2011, SIDMA). They put a stronger condition on the copies of G, that they be embedded isometrically. This holds as well for nearly all of our constructions.