International Workshop On Combinatorial Algorithms
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# Finding posets $P$ in a family of subsets

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presented at IWOCA 2013
updated on 29 August, 2014

We present some problems concerned with finding copies of a given finite poset $P$ in a family $F$ of subsets of an $n$-set $[n] := \{1, \ldots, n\}$. We say $P$ is a (weak) subposet of $F$ if there exists an injection of $P$ into $F$ that preserves order, meaning that if $x < y$ in $P$, then $f(x) < f(y)$ in the Boolean lattice $B_n$ of all subsets of $[n]$ ordered by inclusion, that is $f(x)$ is a subset of $f(y)$. If, moreover, $f(x)$ and $f(y)$ are unrelated whenever $x$ and $y$ are, we say $P$ is an induced (strong) subposet of $F$.

First, we ask for the complexity of determining the minimum $n$ such that $P$ is an induced subposet of $B_n$. Denote this minimum $n$ by $d_2(P)$, called the *2-dimension* of $P$ (Griggs, Stahl, Trotter 1984 [1]). Is it NPC to answer this question: Given $P$ and $n$, is $P$ an induced subposet of $B_n$?

Second, given poset $P$, as well as integers $n \geq k$, what is the complexity of determining whether $P$ is a (weak) subposet of the $k$ middle levels of $B_n$? How hard is it to determine $e(P)$, which we define to be the maximum $k$ such that for all $n$, $P$ is not a subposet of the middle $k$ levels of $B_n$? The parameter $e(P)$ is conjectured (Griggs, Lu 2009 [2]) to determine the asymptotic maximum number of subsets in a $P$-free family of subsets of $[n]$.

Third, it is easy to construct a family of $|P|$ subsets such that $P$ is isomorphic to the family. However, what about constructing a "convex" family of subsets $F$ that contains $P$? Convex means that whenever there are sets $A \subset B \subset C$ with $A, C \in F$, then $B$ is also in $F$. Given a poset $P$ and integers $c$ and $n$, what is the complexity of the question: Is there a convex family $F$ of $c$ subsets in $B_n$ such that $P$ is an induced subposet of $F$? Let $c = c(P)$ denote the minimum $c$ such that there is a convex family $F$ of $c$ subsets containing an induced $P$ (for some $n$). For example, when $P$ is a chain of $k$ elements, then $c(P) = 2^{k-1}$. We wonder how hard it is to determine $c(P)$. This relates to determining asymptotically the maximum number of unrelated copies of $P$ contained in $B_n$ (Dove, Griggs 2013 [3]).

Some examples are listed in the following table, where we use the standard notation $P_k$ (path on $k$ elements), $D_k$ (diamond with $k$ points in the middle, i.e. altogether $k+2$ elements), and $J$ (which is $D_2$ with one edge removed, so 4 elements).

| | $P_3$ | $D_2$ | $D_3$ | $J$ |
|---|---|---|---|---|
| $d_2$ | 2 | 2 | 3 | 3 |
| $e$ | 2 | 2 | 3 | 2 |
| $c$ | 4 | 4 | 8 | 5 |

## References


[1] Jerrold R. Griggs, Jürgen Stahl, and William T. Trotter Jr. A Sperner Theorem on Unrelated Chains of Subsets. *J. of Combinatorial Theory, Series A*, 36:124–127, 1984.

[2] Jerrold R. Griggs and Linyuan Lu. On families of subsets with a forbidden subposet. *Combinatorics, Probability, and Computing*, 18:731–748, 2009.

[3] Andrew P. Dove and Jerrold R. Griggs. In preparation.