The complexity of covering a ladder using cycles

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The ladder graph of length $n$, $L_n$, is the cartesian product $P_2 \square P_n$. We note that $L_n$ contains $\binom{n}{2}$ distinct cycles. We define the following decision problem:

**Ladder Cycle Cover**
INSTANCE: An integer $n$, an integer $m \leq \binom{n}{2}$, an integer $k \leq m$, and a set $C$ of $m$ cycles of the ladder graph $L_n$.
QUESTION: Is there a set $C' \subseteq C$ of $k$ cycles such that the union of all cycles of $C'$ covers $L_n$, i.e. $\bigcup_{C_i \in C'} E(C_i) = E(L_n)$?

**Open problem 1** What is the complexity of the decision problem Ladder Cycle Cover? What if we restrict it to instances having $|C| = n$?

This problem, together with further discussions, is also mentioned in my PhD dissertation [2, Section 8.1]. Let us make a few observations.

**Observation 1** Of course, the answer to the question can be “YES” only if $C$ itself is a valid solution, i.e. $\bigcup_{C_i \in C} E(C_i) = E(L_n)$. This can be checked easily beforehand.

**Observation 2** Since each cycle covers exactly two “step edges” of $L_n$ (the ones coming from the $P_2$ fiber), we need at least $\frac{n}{2}$ cycles in the solution.

**Observation 3** (1) If we were asking only to cover the edges of $L_n$ coming from the $P_n$ fiber, the problem would correspond to Dominating Set in some interval graph, which is solvable in linear time [2].

(2) If we were asking only to cover the edges of $L_n$ coming from the $P_2$ fiber (“step edges”), the problem would correspond to Edge Cover in a graph, which is solvable in polynomial time by a maximum matching approach [3] (cited in [3, Problem GT1]).
Note that we can equivalently state Ladder Cycle Cover as follows:

**Interval And Endpoints Cover**

**INSTANCE:** An integer $n$, an integer $m \leq \binom{n}{2}$, an integer $k \leq m$, and a set $\mathcal{C} = \{s_1, t_1\}, \ldots, \{s_m, t_m\}$ of $m$ pairs from the set $\{1, \ldots, n\}$.

**QUESTION:** Is there a set $\mathcal{C}' \subseteq \mathcal{C}$ of size $k$ such that for each element $i$ of $\{1, \ldots, n\}$:

1. if $i < m$, the interval $[i, i + 1]$ is included in some interval $[s_i, t_i]$ defined by a pair in $\mathcal{C}'$, and
2. $i$ belongs to some pair of $\mathcal{C}'$?

**References**


