Counting numerical semigroups by genus

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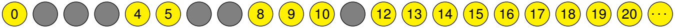
Basic notions

Numerical semigroups

A numerical semigroup is a subset Λ of $\mathbb{N}_0$ satisfying

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $\#(\mathbb{N}_0 \setminus \Lambda)$ is finite (genus$:=g:=\#(\mathbb{N}_0 \setminus \Lambda)$)



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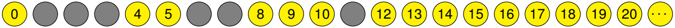

Gaps: $\mathbb{N}_0 \setminus \Lambda$, non-gaps: Λ.

Frobenius number: Largest gap.

Generators

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps.

If $a_1, \dots, a_l$ are the generators of a semigroup Λ then

$$\Lambda = \{n_1 a_1 + \cdots + n_l a_l : n_1, \dots, n_l \in \mathbb{N}_0\}$$

So, $a_1, \dots, a_l$ are necessarily coprime.

If $a_1, \dots, a_l$ are coprime we define the semigroup generated by $a_1, \dots, a_l$ as

$$\langle a_1, \dots, a_n \rangle := \{n_1 a_1 + \cdots + n_l a_l : n_1, \dots, n_l \in \mathbb{N}_0\}.$$

The problem of counting by genus

Counting semigroups by genus

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- $n_1 = 1$, since the unique numerical semigroup of genus 1 is $\mathbb{N}_0 \setminus \{1\}$
- $n_2 = 2$. Indeed the unique numerical semigroups of genus 2 are

$$\{0, 3, 4, 5, \dots\},$$

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Counting semigroups by genus

Conjecture

[Bras-Amorós, 2008]

1. $n_g \geqslant n_{g-1} + n_{g-2}$
2. - $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$
   - $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$

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- Upper and lower bounds for n_g
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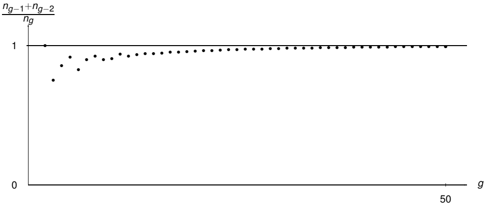
Weaker unsolved conjecture

n_g is increasing.

Counting semigroups by genus

Behavior of $\frac{n_{g-1}+n_{g-2}}{n_g}$



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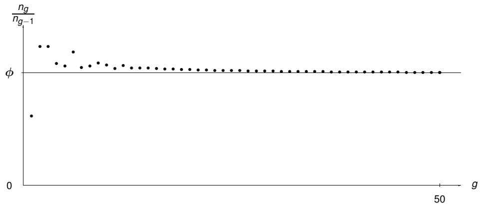
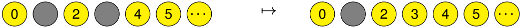


Tree $\mathbb{T}$ of numerical semigroups

From genus g to genus $g-1$

A semigroup of genus g together with its Frobenius number is another semigroup of genus $g-1$.



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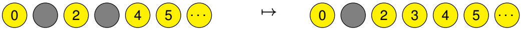
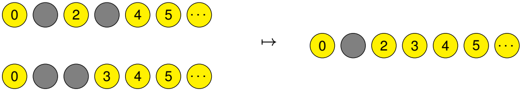


A set of semigroups may give the same semigroup when adjoining their Frobenius numbers.



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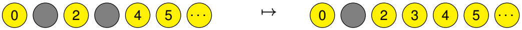

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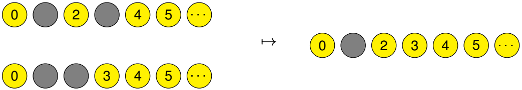
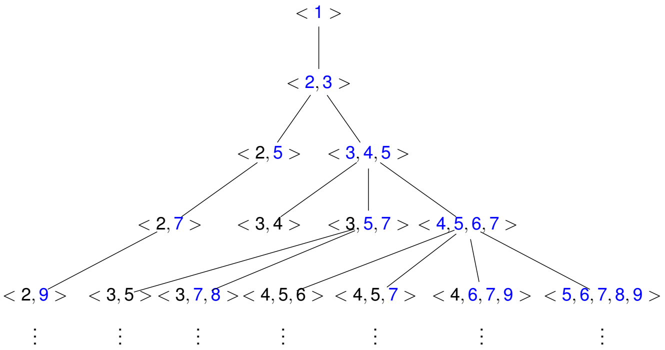


From genus $g-1$ to genus g

All semigroups giving Λ when adjoining to them their Frobenius number can be obtained from Λ by taking out one by one all generators of Λ larger than its Frobenius number.

Tree of numerical semigroups



The descendants of a semigroup are obtained taking away one by one all generators larger than its Frobenius number.

The parent of a semigroup Λ is Λ together with its Frobenius number.
[Rosales, García-Sánchez, García-García, Jiménez-Madrid, 2003]