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Derangements with a Fixed Number of Inversions

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For a permutation $\pi = \pi_1\pi_2\dots\pi_n$, a pair of elements (π_i, π_j) forms an *inversion* if $i < j$ and $\pi_i > \pi_j$. *Mahonian number* $M_{n,k}$ equals the number of permutations of order n with k inversions. Mahonian numbers (sequence A008302 in the OEIS [2]) have the generating function [1]

$$\sum_{k\geq 0} M_{n,k}\cdot x^k = (1+x)\cdot(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}).$$

The total number of inversions in all permutations of order $n > 1$ (sequence A001809 in the OEIS) is given by

$$\binom{n}{2}^2\cdot(n-2)! = \frac{n(n-1)n!}{4}.$$

A permutation $\pi = \pi_1\pi_2\dots\pi_n$ is called *derangement* if $\pi_i \neq i$ for all $i = 1, 2, \dots, n$. The total number of derangements of order n (sequence A000166 in the OEIS) is given by the formula [1]

$$D_n = \sum_{i=0}^{n}(-1)^i\frac{n!}{i!}.$$

The total number T_n of inversions in all derangements of order n (sequence A216239 in the OEIS) can be computed as

$$T_n = \frac{(3n^2-n+1)\cdot D_n + (-1)^n\cdot(n-1)}{12}.$$

Let $N_{n,k}$ be the number of derangements of order n with k inversions (sequence A228924 in the OEIS). It follows that for any $n > 1$,

$$\sum_{k=0}^{n(n-1)/2} N_{n,k} = D_n$$

and

$$\sum_{k=0}^{n(n-1)/2} k \cdot N_{n,k} = T_n.$$

Apparently, no explicit formula for $N_{n,k}$ is known. I propose finding such a formula as a problem for the IWOCA 2016 Open Problem Session.

References

[1] Stanley, R.P.: Enumerative Combinatorics, vol. 1. Cambridge University Press, New York, NY, 1997.

[2] The OEIS Foundation: The On-Line Encyclopedia of Integer Sequences. Published electronically at `http://oeis.org`, 2016.