Constructing Antidictionaries in Output-Sensitive Space

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Definition

A word v is an absent word of some word w if v does not occur as a factor in w.

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Definition

The set $\mathcal{M}_w$ of MAWs of w is called the antidictionary of w.

Applications of Minimal Absent Words

Antidictionaries are used in many real-world applications:

- Data compression (e.g., on-line lossless compression)
- Sequence comparison (e.g., alignment-free sequence comparison)
- Pattern matching (e.g., on-line string matching)
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Most of the times, a reduced antidictionary $\mathcal{M}^{\ell}$ is considered, consisting of those MAWs whose length is bounded by some threshold ℓ.

Properties of Minimal Absent Words

The theory of MAWs is well developed. For example, it is know that:

Theorem

1. *A word of length n has $\mathcal{O}(n)$ different MAWs, which can be stored occupying $\mathcal{O}(n)$ total space.*
2. *One can compute the antidictionary of a word of length n in $\mathcal{O}(n)$ time and space.*
3. *Any word of length n can be reconstructed in $\mathcal{O}(n)$ time and space from its (complete) antidictionary.*
4. *The maximal length of a MAW equals 2 + the maximal length of a repeated factor. Thus, for a random[a] word of length n, the longest MAW has length $\Theta(\log_{|\Sigma|} n)$.*

[a] generated by a Bernoulli i.i.d. source

Algorithms for Computing Minimal Absent Words

There exist several efficient algorithms for computing the (reduced) antidictionary of a word of length n, e.g.:

- $O(n)$ time and space using a global data structure (e.g., SA) [Barton, Héliou, Mouchard, Pissis, 2014]

  — can be executed in external memory [Héliou, Pissis, Puglisi, 2017]

- $O(n) + |\mathcal{M}^\ell|$ time using $O(\min\{n, \ell z\})$ space, where z is the size of the LZ77 factorization, using the truncated DAWG [Fujishige, Takuya, Diptarama, 2018]

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However, all these algorithms require $\Omega(n)$ space due to the construction of a global data structure on the input word.

Number and Distribution of Minimal Absent Words

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Problem

Compute the (reduced) antidictionary in output-sensitive space.

Strategy

Idea:

- Divide the input word y into k words each of which, alone, fits in the internal memory, with a suitable overlap of length ℓ so as not to lose information.

$$y = y_1 \# y_2 \# \cdots \# y_k, \qquad \# \notin \Sigma$$

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Formally, we state the following

Problem

Given k words $y_1, y_2, \ldots, y_k$ over an alphabet Σ and an integer $\ell > 0$, compute the set $\mathcal{M}^{\ell}_{y_1\#\ldots\#y_k}$ of minimal absent words of length at most ℓ of $y = y_1\#y_2\#\ldots\#y_k$, $\# \notin \Sigma$.

Theoretical Results

Here is an illustration of the theoretical setting: Let $y = y_1 \# y_2$.

We are allowed to store in internal memory y_1 and y_2 but not y.

Our goal is to compute $\mathcal{M}_y^\ell$ from $\mathcal{M}_{y_1}^\ell$ and $\mathcal{M}_{y_2}^\ell$.

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Let $x \in \mathcal{M}_y^\ell$. We separate two cases:

1. x belongs to $\mathcal{M}_{y_1}^\ell \cup \mathcal{M}_{y_2}^\ell$ (Case 1)
2. x does not belong to $\mathcal{M}_{y_1}^\ell \cup \mathcal{M}_{y_2}^\ell$ (Case 2)

Theoretical Results

Lemma (Case 1)

A word $x \in \mathcal{M}^{\ell}_{y_1}$ (resp. $x \in \mathcal{M}^{\ell}_{y_2}$) belongs to $\mathcal{M}^{\ell}_{y}$ if and only if x is a superword of a word in $\mathcal{M}^{\ell}_{y_2}$ (resp. in $\mathcal{M}^{\ell}_{y_1}$).

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Example

Let $y_1 = \text{abaab}$, $y_2 = \text{bbaaab}$ and $\ell = 5$. $y = \text{abaab\#bbaaab}$. We have $\mathcal{M}^{\ell}_{y_1} = \{\texttt{bb,aaa,bab,aaba}\}$ and
$\mathcal{M}^{\ell}_{y_2} = \{\texttt{bbb,aaaa,baab,aba,bab,abb}\}$.
The word bab is contained in $\mathcal{M}^{\ell}_{y_1} \cap \mathcal{M}^{\ell}_{y_2}$ so it belongs to $\mathcal{M}^{\ell}_{y}$. The word aaba $\in \mathcal{M}^{\ell}_{y_1}$ is a superword of aba $\in \mathcal{M}^{\ell}_{y_2}$ hence aaba $\in \mathcal{M}^{\ell}_{y}$. On the other hand, the words bbb, aaaa and abb are superwords of words in $\mathcal{M}^{\ell}_{y_1}$, hence they belong to $\mathcal{M}^{\ell}_{y}$. The remaining MAWs are not superwords of MAWs of the other word.

$$\mathcal{M}^{\ell}_{y} \cap (\mathcal{M}^{\ell}_{y_1} \cup \mathcal{M}^{\ell}_{y_2}) = \{\texttt{aaaa,bab,aaba,abb,bbb}\}.$$

Theoretical Results

We define the *reduced sets* of MAWs, $\mathcal{R}^{\ell}_{y_i}$, as those sets obtained from $\mathcal{M}^{\ell}_{y_i}$ after removing those words that are superwords of a word in $\mathcal{M}^{\ell}_{y_j}$, $\{i, j\} = \{1, 2\}$.

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Lemma (Case 2)

Let $x \in \mathcal{M}^{\ell}_{y} \setminus (\mathcal{M}^{\ell}_{y_1} \cup \mathcal{M}^{\ell}_{y_2})$. *Then* x *has a prefix* x_i *in* $\mathcal{R}^{\ell}_{y_i}$ *and a suffix* x_j *in* $\mathcal{R}^{\ell}_{y_j}$, *for* i, j *such that* $\{i, j\} = \{1, 2\}$.

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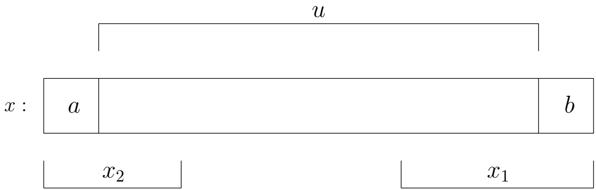
Example

Let $y_1 =$ abaab and $y_2 =$ bbaaab. $y =$ abaab#bbaaab. We have $\mathcal{R}^{\ell}_{y_1} = \{bb,aaa\}$ and $\mathcal{R}^{\ell}_{y_2} = \{$baab,aba$\}$.

Consider $x =$ abaaa $\in \mathcal{M}^{\ell}_{y} \setminus (\mathcal{M}^{\ell}_{y_1} \cup \mathcal{M}^{\ell}_{y_2})$ (Case 2 MAW).

There is a MAW$x_2 \in \mathcal{R}^{\ell}_{y_2}$ that is a prefix of abaa and this is aba. Analogously, there is an $x_1 \in \mathcal{R}^{\ell}_{y_1}$ that is a suffix of abaaa and this is aaa.

Theoretical Results



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Theoretical Results

We come to the following general result, which is the theoretical basis of our algorithm:

Theorem

Let $N > 1$, and let $x \in \mathcal{M}^{\ell}_{y_1\#\ldots\#y_N}$. Then, either $x \in \mathcal{M}^{\ell}_{y_1\#\ldots\#y_{N-1}} \cup \mathcal{M}^{\ell}_{y_N}$ (Case 1 MAWs) or, otherwise, $x \in \mathcal{M}^{\ell}_{y_i\#y_N} \setminus (\mathcal{M}^{\ell}_{y_i} \cup \mathcal{M}^{\ell}_{y_N})$ for some i. Moreover, in this latter case, x has a prefix in $\mathcal{R}^{\ell}_{y_1\#\ldots\#y_{N-1}}$ and a suffix in $\mathcal{R}^{\ell}_{y_N}$, or the converse, i.e., x has a prefix in $\mathcal{R}^{\ell}_{y_N}$ and a suffix in $\mathcal{R}^{\ell}_{y_1\#\ldots\#y_{N-1}}$ (Case 2 MAWs).

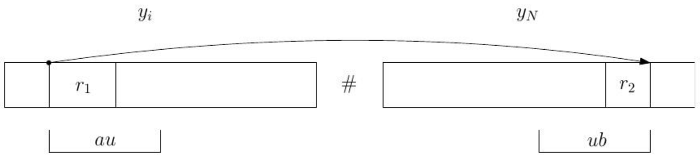
The Algorithm

At the Nth step, we have in memory the set $\mathcal{M}^{\ell}_{y_1\#\ldots\#y_{N-1}}$. Our algorithm works as follows:

1. We read word y_N from the disk and compute $\mathcal{M}^{\ell}_{y_N}$ in time $\mathcal{O}(|y_N|)$.
2. We compute Case 1 MAWs using the first Lemma.
3. For every $i \in [1, N-1]$, we perform the following to compute Case 2 MAWs:

The Algorithm

1. Read word y_i from the disk. Construct the suffix tree T_x of word $x = y_i \# y_N$ in time $\mathcal{O}(|y_i| + |y_N|)$. Use T_x to locate all occurrences of elements of $\mathcal{R}^{\ell}_{y_N}$ in y_i.
2. Compute the set $\mathcal{M}^{\ell}_{y_i \# y_N}$ and output the words.
3. Suppose au occurs in y_i and ub in y_N. Check whether au starts where a word r_1 of $\mathcal{R}^{\ell}_{y_N}$ starts and ub ends where a word r_2 of $\mathcal{R}^{\ell}_{y_1 \# \ldots \# y_{N-1}}$ ends. If this is the case and $|u| \geq \max\{|r_1|, |r_2|\} - 1$, then aub is added to our output set M, otherwise discard it. The case when au occurs in y_N and ub in y_i is treated analogously.



The Algorithm

Let $\textsc{MaxIn}$ be the length of the longest word in $\{y_1, \ldots, y_k\}$ and $\textsc{MaxOut} = \max\{|| \mathcal{M}^{\ell}_{y_1\#\ldots\#y_N} || : N \in [1, k]\}$.

Theorem

Given k words $y_1, y_2, \ldots, y_k$ and an integer $\ell > 0$, all $\mathcal{M}^{\ell}_{y_1}, \ldots, \mathcal{M}^{\ell}_{y_1\#\ldots\#y_k}$ can be computed in $\mathcal{O}(kn + \sum_{N=1}^{k} || \mathcal{M}^{\ell}_{y_1\#\ldots\#y_N} ||)$ total time using $\mathcal{O}(\textsc{MaxIn} + \textsc{MaxOut})$ space, where $n = |y_1\# \ldots \#y_k|$.

Proof-of-Concept Experiments

The algorithm has been implemented in the `C++` programming language.
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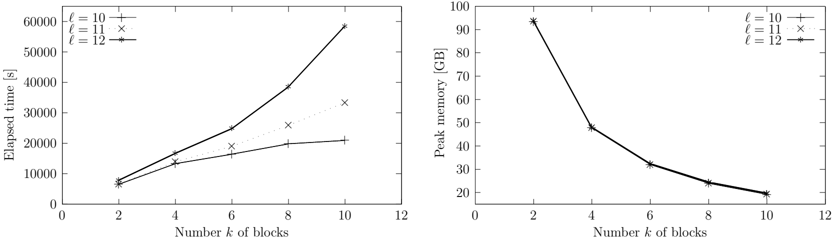
We ran the program by splitting the genome into $k = 2, 4, 6, 8, 10$ blocks and setting $\ell = 10, 11, 12$.

Proof-of-Concept Experiments

The figure depicts the change in elapsed time and peak memory usage as k and ℓ increase (space-time tradeoff).

Graph (a) shows an increase of time as k and ℓ increase. Graph (b) shows a decrease in memory as k increases.



(a) **(b)**

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3. Our technique could serve as a basis for a new parallelisation scheme for constructing antidictionaries, in which several blocks are processed concurrently.

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Thank you!