Introduction to Fuzzy Inference System

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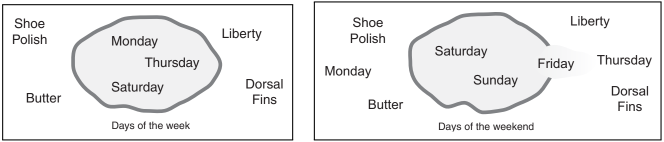
Fuzzy Logic:

- Can a computer compute with words?
- Expert can solve a lot of complex problems using imprecision such as common sense and expert knowledge.
- Common sense and expert knowledge can be represented by linguistic rules, say, in *If-Then* format.
- Fuzzy logic is the theory of fuzzy sets, which is used to handle fuzziness/imprecision/ambiguity/vagueness.
- Fuzzy set theory which can mimic the human spirit for approximation reasoning based on imprecise information.
- By using fuzzy logic, “human spirit” can be computed/represented mathematically.
- Serves as a structure for represent knowledge and learning using natural inspired learning algorithms

Classical Sets and Fuzzy Sets:

- How do we represent imprecision and vagueness?
- How do you understand the phrase “Today is Weekend”?

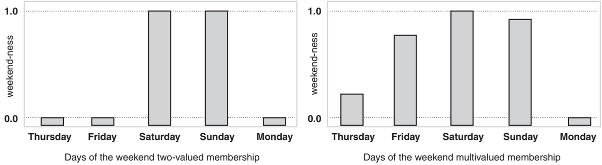


(h) Classical set. (i) Fuzzy set.

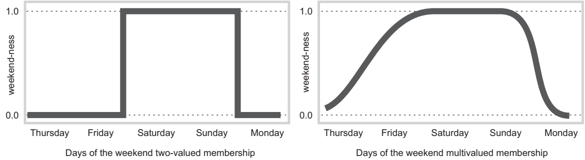
Figure 6: Classical and fuzzy sets for weekend (diagram from Matlab).

Classical Sets and Fuzzy Sets:

- Use *membership functions* to measure the degree (*membership grade*).
- How do you understand the phrase “The current season is Summer”?



(a) Discrete membership function.



(b) Continuous membership functions.

Figure 7: Discrete and continuous membership functions (diagram from Matlab).

Classical Sets and Fuzzy Sets:

- Multiple *fuzzy sets* where *membership functions* are with different labels (*fuzzy terms* or *linguistic terms*).

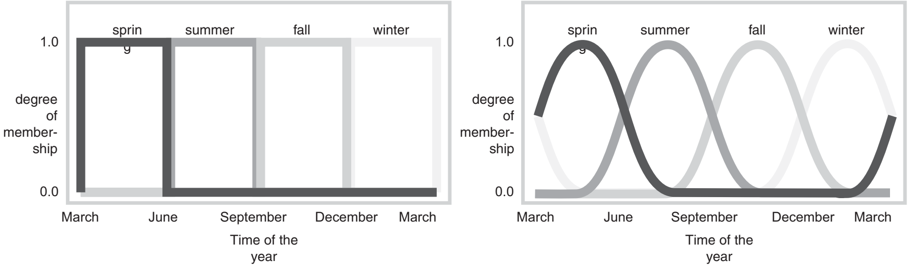


Figure 8: Fuzzy sets with different labels (diagram from Matlab).

Driving problem: I am driving and want to keep a safety distance between cars. When the distance from the front car is x, what speed should I keep?

Linguistic Rules:

Rule 1: **If** distance is *small* **Then** speed is *low*

Rule 2: **If** distance is *medium* **Then** speed is *steady*

Rule 3: **If** distance is *large* **Then** speed is *high*

More specific question*: When the distance from the front car is 3.5 m or so, what speed should I keep?*

Remark: Different people have different meaning of *small*, *medium*, *large*, *high*, *steady* and *low*. How do you define them? How do you explain these terms to computers?

Linguistic Rules:

Rule 1: **If** distance is *small* **Then** speed is *low*

Rule 2: **If** distance is *medium* **Then** speed is *steady*

Rule 3: **If** distance is *large* **Then** speed is *high*

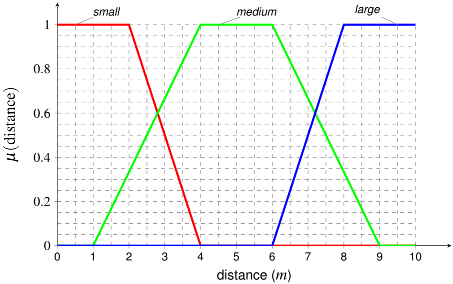


Figure 9: Fuzzification Process.

***More specific question**: When the distance from the front car is 3.5 m or so, what speed should I keep? **My Answer**: The speed should be not very "low", more toward "steady" but definitely not "high".*

Linguistic Rules:

Rule 1: **If** distance is *small* **Then** speed is *low*

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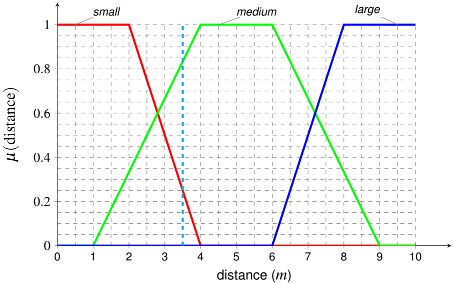

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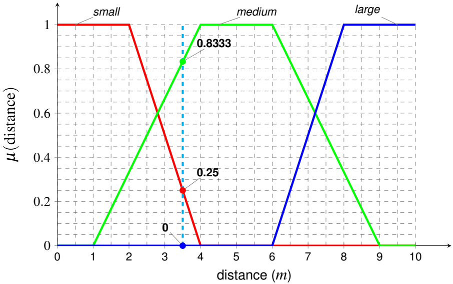


Figure 9: Fuzzification Process.

***More specific question**: When the distance from the front car is 3.5 m or so, what speed should I keep? **My Answer**: The speed should be not very "low", more toward "steady" but definitely not "high".*

Fuzzy Inference System

- A **fuzzy inference system** (FIS) is also known as *fuzzy-rule-based system*, *fuzzy expert system*, *fuzzy logic system*, *fuzzy model*, *fuzzy associative memory* (FAM) and *fuzzy logic controller* and *fuzzy system*.
- An FIS is a computing framework based on the concepts of fuzzy set theory, fuzzy (If-Then) rules and fuzzy reasoning.
- An FIS consists of **4 components**: *fuzzifier*, *Knowledge base* (rule base or database), *fuzzy inference engine* and *defuzzifier*.

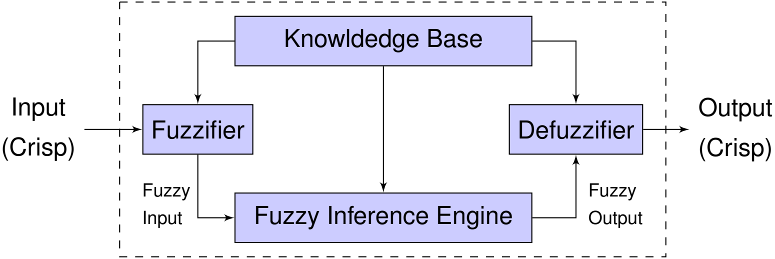
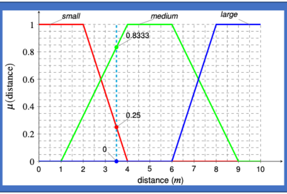


Figure 10: A diagram of fuzzy inference system.

- **Fuzzifiers:** It maps the crisp (real-valued) input into a fuzzy set defined in the universe of discourse (the domain of the fuzzy set) $\mathbf{X}$ characterised by membership functions. This process is called *fuzzification*. Note: The input can also be a fuzzy set.
- **Knowledge Base:** It is a database consisting of linguistic rules in If-Then format.
- **Fuzzy Inference Engine:** Using the If-Then rules in *Knowledge base*, it performs reasoning by producing a fuzzy output according to the fuzzy input given by the *fuzzifier*.
- **Defuzzifiers:** It converts the fuzzy output given by the *fuzzy inference engine* to produce a crisp (real-valued) output. This process is called *defuzzifiaction*.

Fuzzy Inference System

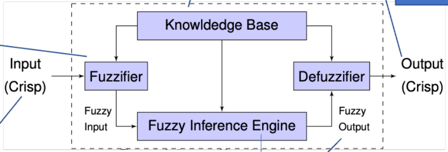


Rule 1: **If** distance is *small* **Then** speed is *low*

Rule 2: **If** distance is *medium* **Then** speed is *steady*

Rule 3: **If** distance is *large* **Then** speed is *high*

Output:
20 miles/hour

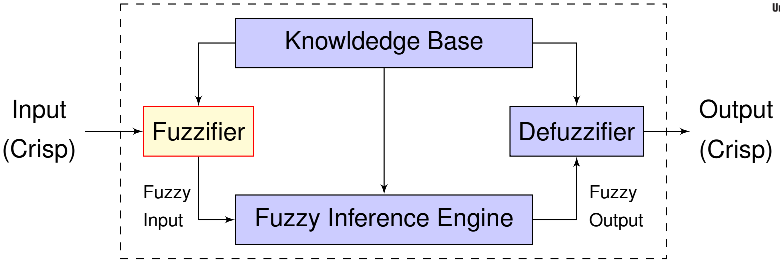


Input: 3.5m

Level of Distance	Speed
0.25 of Small	Not very Low
0.8333 of Medium	More Steady
0 of Large	Definitely not High

Speed should not be very Low, more towards Steady but definitely not High

Fuzzifiers



Knowldedge Base
Input
(Crisp)
Fuzzifier
Defuzzifier
Output
(Crisp)
Fuzzy
Input
Fuzzy Inference Engine
Fuzzy
Output

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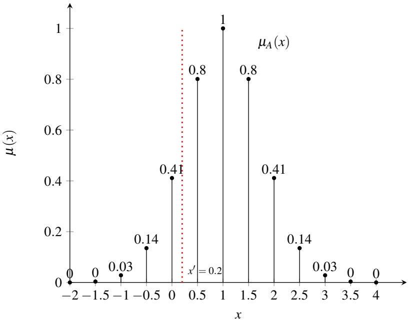
- The input is turned to fuzzy sets thought membership functions.
- A fuzzy set is represented by a membership functions (associated with a label called *fuzzy term* or *linguistic term*).

Property of membership functions:

- A membership function can be *discrete* or *continuous*.
- A membership function denoted by $\mu_A(x)$ corresponding to fuzzy set A (A is the *fuzzy term* or *linguistic term*) is characterised by a linear/nonlinear function of premise variable x.
  - E.g., "**If** x is *Positive* **Then** y is *Fast*"; "**If** distance is *small* **Then** speed is *low*"
- It is in the range of $0 \leq \mu_A(x) \leq 1$.
- Considering a particular reading, say, x', $0 \leq \mu_A(x') \leq 1$ is called the *membership degree/grade* or *degree/grade of membership*.
  - E.g., $\mu_A(x') = 0.5$ when $x' = 2$; $\mu_{small}(\text{distance}) = 0.8333$ when $\text{distance} = 3.5$

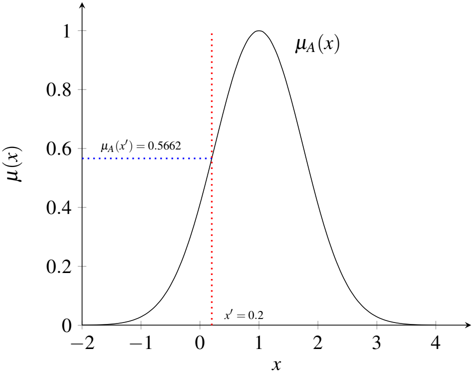
Fuzzifiers

- Notation for discrete fuzzy set: $A = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \cdots \right\} = \left\{ \sum_{x_i \in X} \frac{\mu_A(x_i)}{x_i} \right\}$.
- The horizontal bar is not a quotient but rather a delimiter.
- The summation symbol is not for algebraic summation, but denotes the collection or aggregation of each element. The "+" signs are not the algebraic "add" but are an aggregation or collection operator.
- Example: $A = \left\{ \frac{0}{-2} + \frac{0}{-1.5} + \frac{0.03}{-1} + \frac{0.14}{-0.5} + \frac{0.41}{0} + \frac{0.8}{0.5} + \frac{1}{1} + \frac{0.8}{1.5} + \frac{0.41}{2} + \frac{0.14}{2.5} + \frac{0.03}{3} + \frac{0}{3.5} + \frac{0}{4} \right\}$
- $\mu_A(0.2) = 0.41$



Fuzzifiers

- Notation for continuous fuzzy set: $A = \left\{ \int_X \frac{\mu_A(x)}{x} \right\}$.
- The horizontal bar is not a quotient but rather a delimiter.
- The integral sign is not an algebraic integral but a continuous function-theoretic aggregation operator for continuous variables.



More properties of membership functions:

- *Core:* The core of a membership function for a fuzzy set $\underset{\sim}{A}$ is the region of the universe of which $\mu_{\underset{\sim}{A}}(x) = 1$.
- *Support:* It is defined as the region of the universe of which $\mu_{\underset{\sim}{A}}(x) > 0$.
- *Boundaries:* It is defined as the region of the universe of which $0 < \mu_{\underset{\sim}{A}}(x) < 1$.

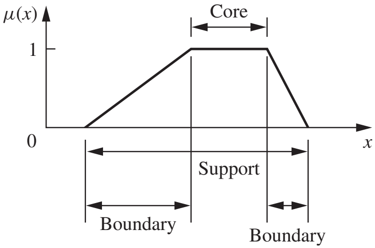


Figure 11: Core, support and boundaries of a fuzzy set.

More properties of membership functions:

- *Normal/subnormal fuzzy set:* A fuzzy set is said to be normal if its membership function has at least one element of x whose membership grade is 1, i.e., $\mu_{\underset{\sim}{A}} = 1$, otherwise, a subnormal fuzzy set.
- *height* of a fuzzy set hgt$(\underset{\sim}{A})$: $\underset{\sim}{A} = \max\{\mu_{\underset{\sim}{A}}\}$ which is the maximum membership degree of a membership function. So, hgt$(\underset{\sim}{A}) = 1 \Rightarrow$ normal membership function; hgt$(\underset{\sim}{A}) < 1 \Rightarrow$ subnormal membership function.

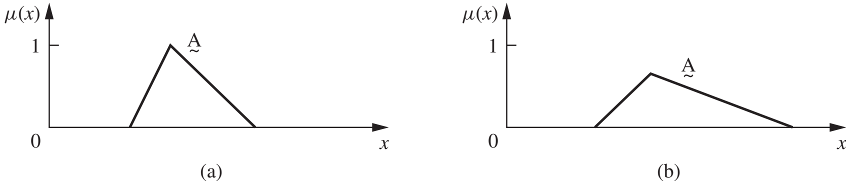


Figure 12: (a) Normal and (b) subnormal fuzzy sets.

More properties of membership functions:

- *Convex/non-convex fuzzy set:* A fuzzy set is said to be convex if membership function are strictly monotonically increasing/decreasing or strictly monotonically increasing and then decreasing, otherwise, a non-convex fuzzy set.

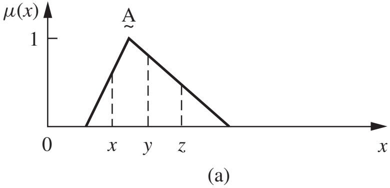


Figure 13: (a) convex and (b) non-convex fuzzy sets.

- Fuzzification is the process turning the crisp input to a fuzzy value (membership grade).

Linguistic Rules:

Rule 1: **If** distance is *small* **Then** speed is *low*

Rule 2: **If** distance is *medium* **Then** speed is *steady*

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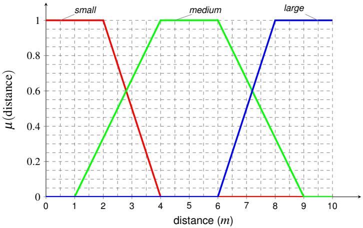

Figure 14: Fuzzification Process.

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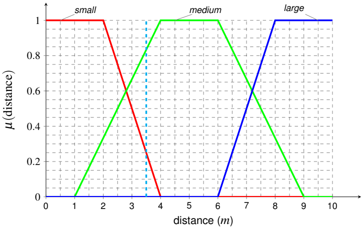

Figure 14: Fuzzification Process.

Fuzzifiers

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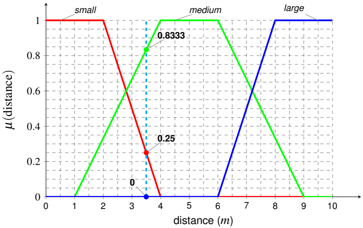


Figure 14: Fuzzification Process.

Common Membership Functions

- **Singleton membership function:** $\mu_{\underset{\sim}{A}}(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & otherwise \end{cases}$

- **Triangular membership function:** $\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{m-a}, & a < x \leq m \\ \frac{b-x}{b-m}, & m < x < b \\ 0, & x \geq b \end{cases}$ where $a \leq m \leq b$

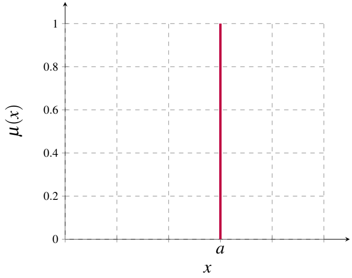


Figure 15: Singleton membership function.

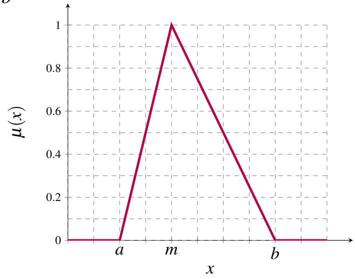


Figure 16: Triangular membership function.

Common Membership Functions

- **Trapezoidal membership function:** $\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0, & x < a \text{ or } x > d \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \frac{d-x}{d-c}, & c \le x \le d \end{cases}$ where $a \le b \le c \le d$

- **Gaussian membership function:** $\mu_{\underset{\sim}{A}}(x) = e^{-\frac{(x-m)^2}{2\sigma^2}}$

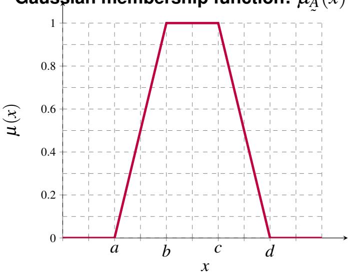


Figure 17: Trapezoidal membership function.

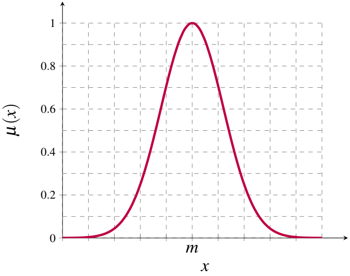


Figure 18: Gaussian membership function.

Common Membership Functions

- **Z-shaped membership function:** $\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0, & x > d \\ \frac{d-x}{d-c}, & c \leq x \leq d \\ 1, & x < c \end{cases}$ where $c \leq d$

- **S-shaped membership function:** $\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$ where $a \leq b$

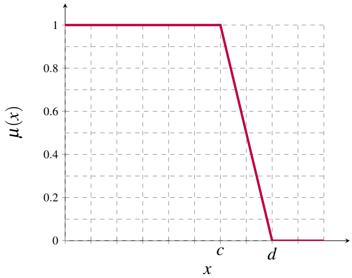


Figure 19: Z-shaped membership function.

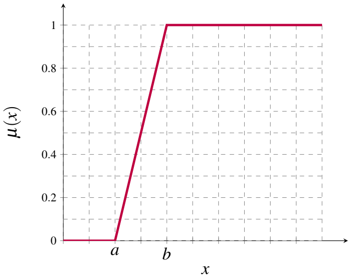
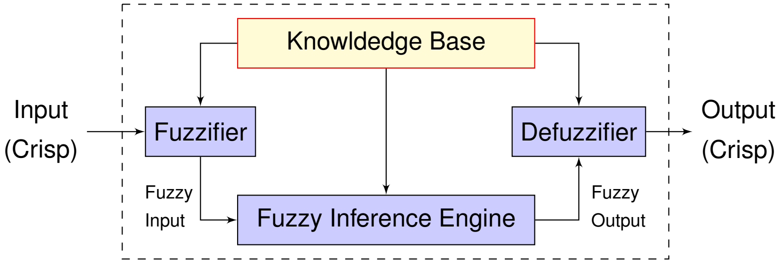


Figure 20: S-shaped membership function.



Recall the example - Linguistic Rules:

Rule 1: **If** distance is *small* **Then** speed is *low*

Rule 2: **If** distance is *medium* **Then** speed is *steady*

Rule 3: **If** distance is *large* **Then** speed is *high*

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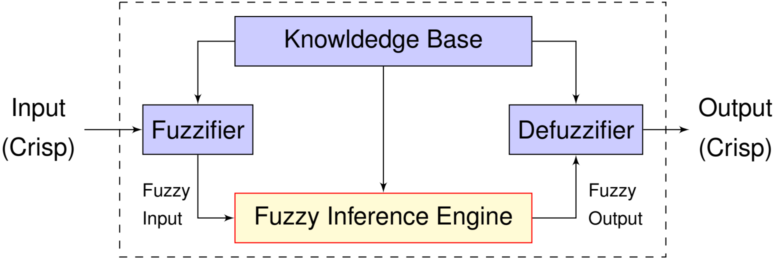
- The knowledge base is the rule base representing the expertise knowledge dealing with a specific problem.
- Linguistic rule: IF premise (antecedent) THEN conclusion (consequent).
- The knowledge base can have more than one rule.

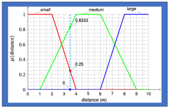
General rule format:

$$\text{Rule } i\text{: } \textbf{IF } \underbrace{x_1 \text{ is } \underset{\sim}{A}_{i1} \text{ and/or } x_2 \text{ is } \underset{\sim}{A}_{i2} \text{ and/or } \cdots}_{antecedent} \textbf{ THEN } \underbrace{y \text{ is } \underset{\sim}{B}_i}_{consequent}, \; i = 1, 2, \ldots, r$$

- $x_1, x_2, \cdots$ are the fuzzy/linguistic variables.
- y is the output of the fuzzy inference system.
- "and' and "or" are fuzzy operators.
- $\underset{\sim}{A}_{i1}, \underset{\sim}{A}_{i2}, \cdots$ are the fuzzy sets (associated with a linguistic variable) representing the i^{th} *antecedent* pairs.
- $\underset{\sim}{B}^i$ is the fuzzy set (associated with a linguistic variable) representing the i^{th} *consequent*.
- r is number of rules.

Fuzzy Inference Engine



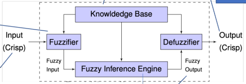


Rule 1: **If** distance is *small* **Then** speed is *low*

Rule 2: **If** distance is *medium* **Then** speed is *steady*

Rule 3: **If** distance is *large* **Then** speed is *high*

Output:
20 miles/hour



Input: 3.5m

Level of Distance	Speed
0.25 of Small	Not very Low
0.8333 of Medium	More Steady
0 of Large	Definitely not High

Speed should not be very Low, more towards Steady but definitely not High

Fuzzy Inference Engine

- **Fuzzy inference engine** is to produce the fuzzy output according to the crisp inputs based on the knowledge (knowledge base) represented by IF-THEN rule. This is the process of reasoning. It generally involves two processes, i.e., *rule evaluation* and *rule aggregation*
  - **Rule evaluation** (implication) is to apply the fuzzy set operators (AND, OR, NOT) to the antecedents to determine the firing strength of each rule.
  - **Rule aggregation** is to combine the output (consequents) fuzzy sets using the firing strengths obtained in the process of *rule evaluation*.
- There are three standard fuzzy set operations
  - *Fuzzy union* operation (*OR*), also known as *t-norm* or conjunction operator.
  - *Fuzzy intersection* operation (*AND*), also known as *t-conorm*, *s-norm* operation, disjunction operation.
  - *Fuzzy complement* operation (*NOT*).

Example - Linguistic Rules (max for "or" and min for "and"; $x = 3.5$, $y = 2$)

Rule 1: **If** x is *small* and y is *negative* **Then** z is *low*

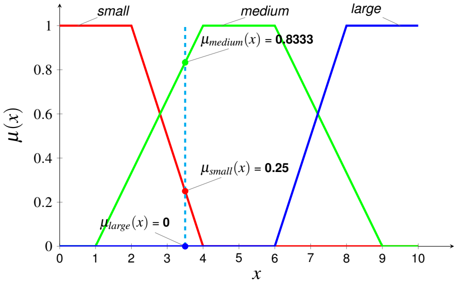
(Rule evaluation: $\min(\mu_{small}(x), \mu_{negative}(y)) = \min(0.8333, 0.0561) = 0.0561$)

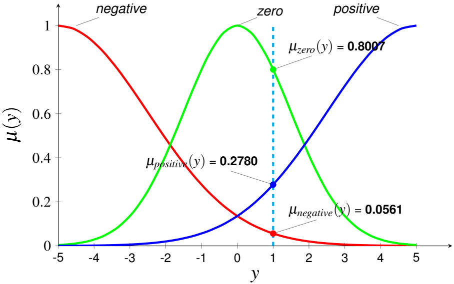
Rule 2: **If** x is *medium* or y is *zero* **Then** z is *middle*

(Rule evaluation: $\max(\mu_{medium}(x), \mu_{zero}(y)) = \max(0.25, 0.8007) = 0.8007$)

Rule 3: **If** x is *large* or y is not *positive* **Then** z is *high*

(Rule evaluation: $\max(\mu_{large}(x), \mu_{not\ positive}(y)) = \max(0, 1 - 0.2780) = 0.7220$)





Fuzzy OR operator (fuzzy union operator)

- *Maximum:* $\mu_{\underset{\sim}{A}\cup\underset{\sim}{B}}(x,y)=\mu_{\underset{\sim}{A}}(x)\vee\mu_{\underset{\sim}{B}}(y)=\max(\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(y)$
- *Algebraic sum:* $\mu_{\underset{\sim}{A}\cup\underset{\sim}{B}}(x,y)=\mu_{\underset{\sim}{A}}(x)+\mu_{\underset{\sim}{B}}(y)-\mu_{\underset{\sim}{A}}(x)\times\mu_{\underset{\sim}{B}}(y)$

Fuzzy AND operator (fuzzy intersection operator)

- *Minimum:* $\mu_{\underset{\sim}{A}\cap\underset{\sim}{B}}(x,y)=\mu_{\underset{\sim}{A}}(x)\wedge\mu_{\underset{\sim}{B}}(y)=\min(\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(y))$
- *Product:* $\mu_{\underset{\sim}{A}\cap\underset{\sim}{B}}(x,y)=\mu_{\underset{\sim}{A}}(x)\times\mu_{\underset{\sim}{B}}(y)$

Fuzzy NOT operator (fuzzy complement operator)

- *Complement:* $\mu_{\overline{\underset{\sim}{A}}}(x)=1-\mu_{\underset{\sim}{A}}(x)$

Fuzzy Inference Engine

Example (rule evaluation - discrete fuzzy sets): Consider the fuzzy sets $small = \left\{ \frac{0}{1} + \frac{0}{2} + \frac{1}{3} + \frac{0}{4} \right\}$ and $negative = \left\{ \frac{0}{1} + \frac{0.5}{2} + \frac{1}{3} + \frac{0.5}{4} + \frac{0}{5} \right\}$, and the following fuzzy rule:

Rule 1: **If** x is *small* and y is *negative* **Then** z is *low*.

Find the firing strength of Rule 1 when $x = 3$ and $y = 2$ where fuzzy "AND" operation is the minimum operator.

Solution:

Firing strength (Rule evaluation):

$$\mu_{small \cap negative}(3,2) = \min(\mu_{small}(3), \mu_{negative}(2)) = \min(1, 0.5) = 0.5$$

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-min

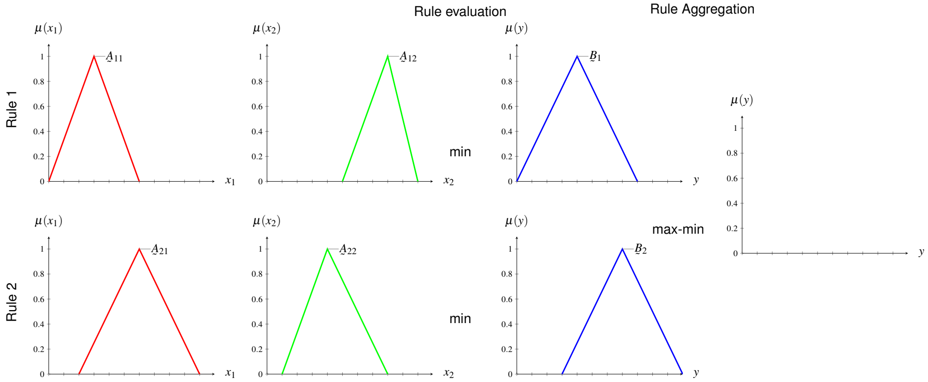


Figure 21: Mamdani (max-min) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-min

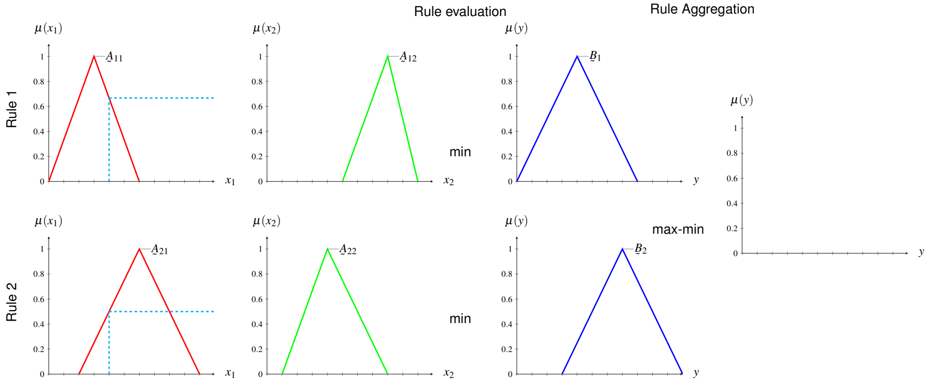


Figure 21: Mamdani (max-min) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-min

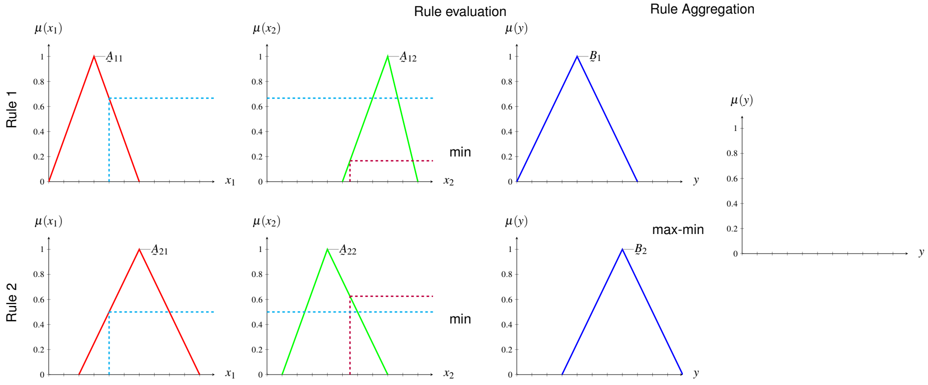


Figure 21: Mamdani (max-min) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

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Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-min

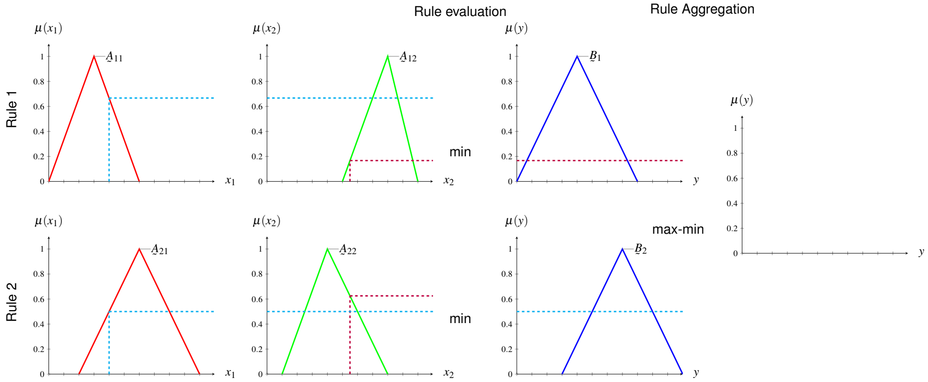

Figure 21: Mamdani (max-min) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-min

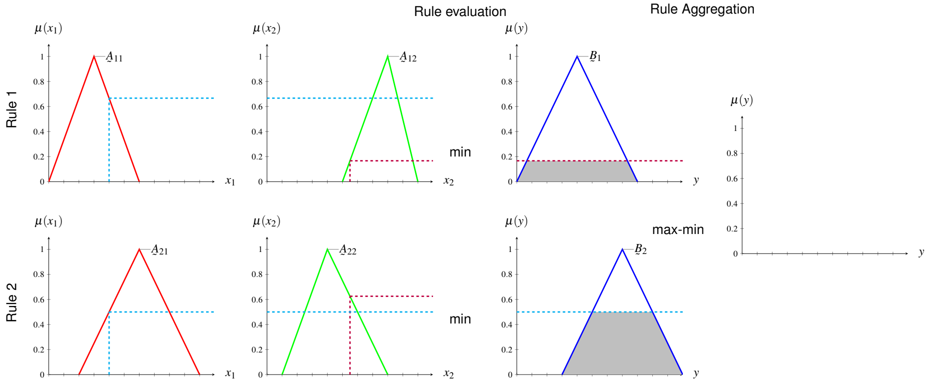

Figure 21: Mamdani (max-min) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-min

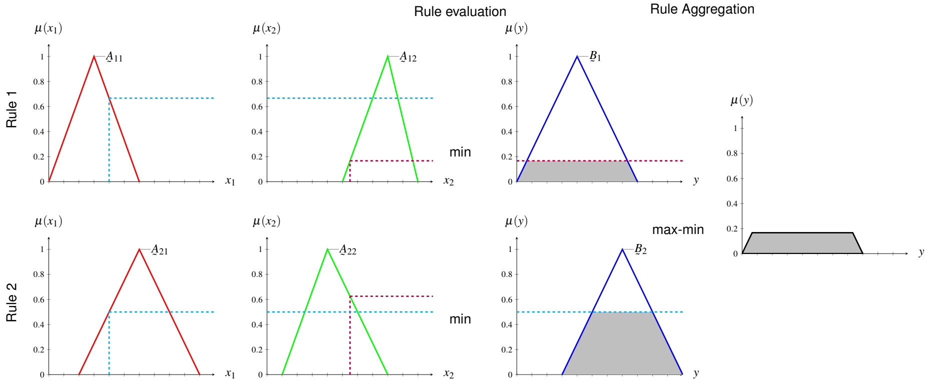

Figure 21: Mamdani (max-min) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-min

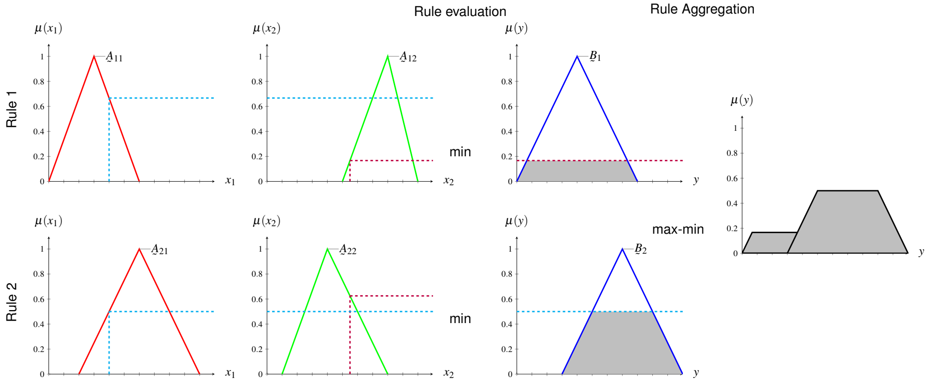

Figure 21: Mamdani (max-min) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-min

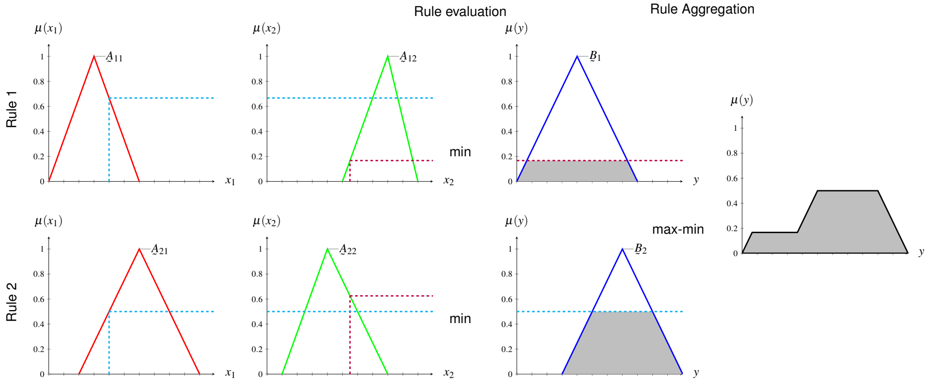

Figure 21: Mamdani (max-min) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-product

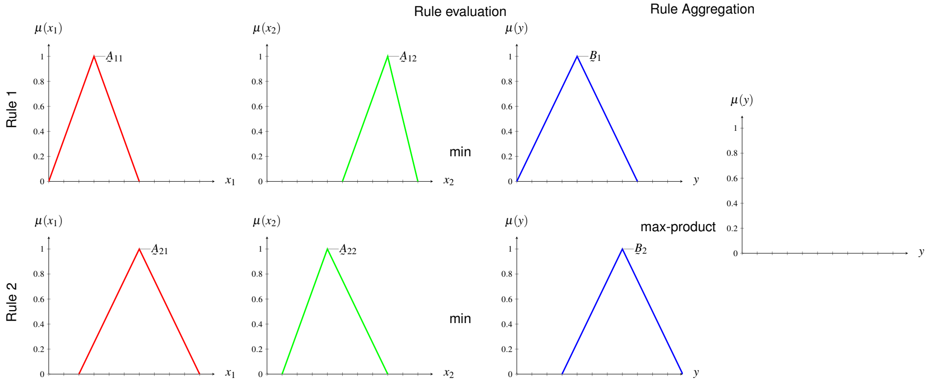

Figure 22: Mamdani (max-product) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-product

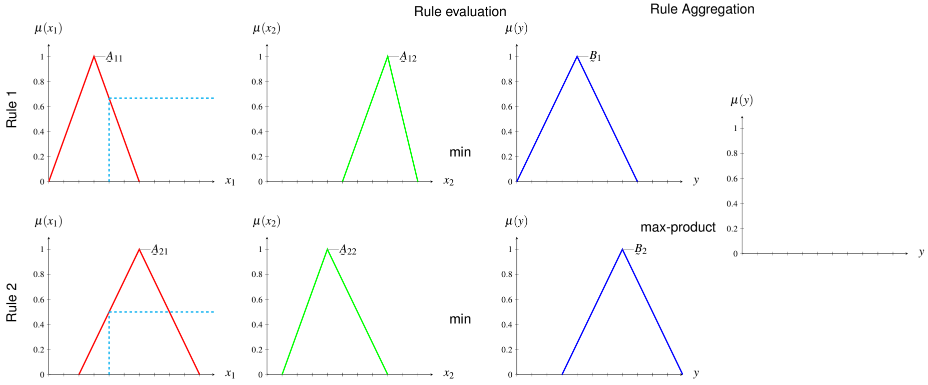

Figure 22: Mamdani (max-product) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-product

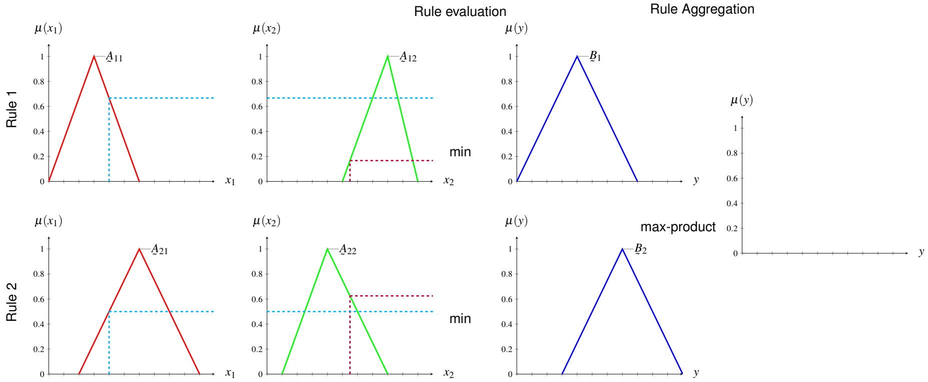

Figure 22: Mamdani (max-product) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-product

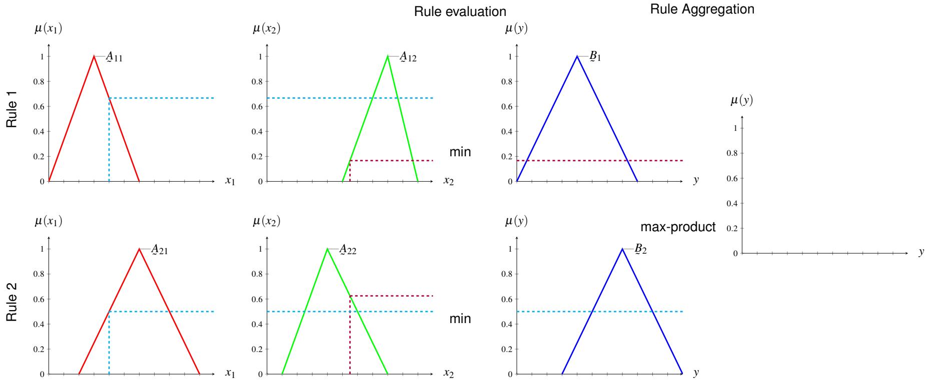

Figure 22: Mamdani (max-product) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-product

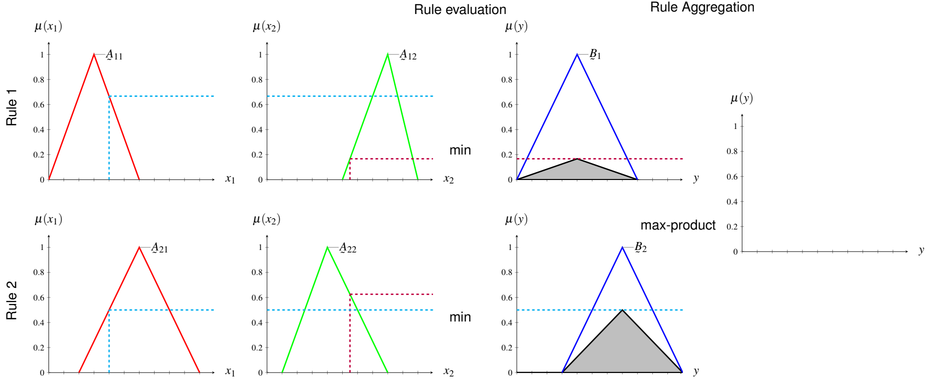

Figure 22: Mamdani (max-product) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-product

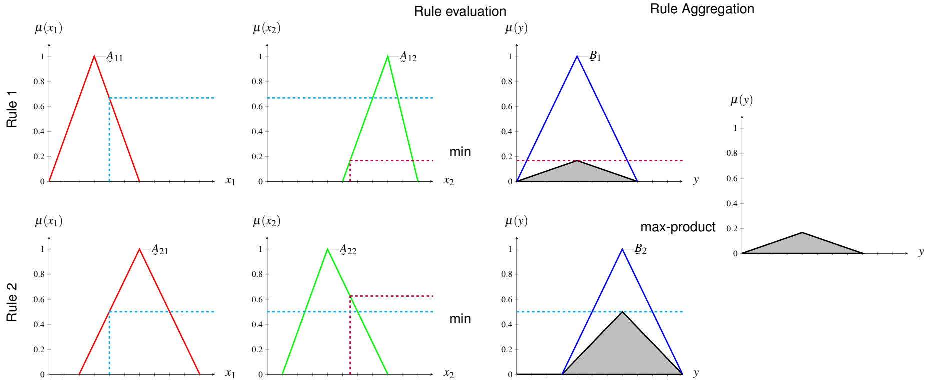

Figure 22: Mamdani (max-product) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-product

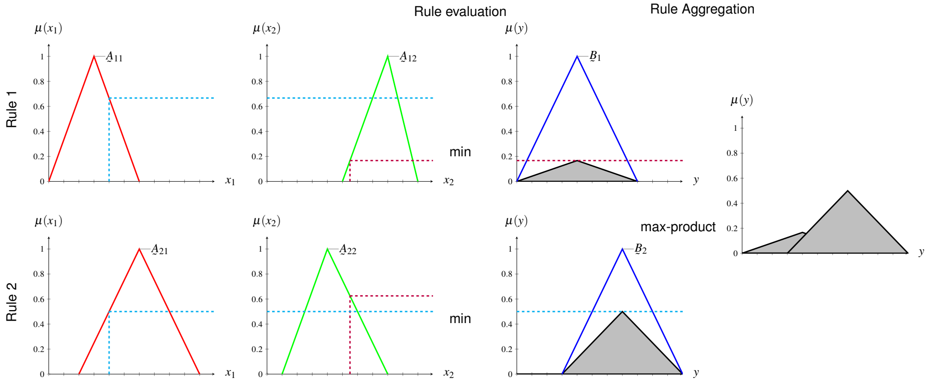

Figure 22: Mamdani (max-product) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

Rule Aggregation: Consider two simple rules:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$(Small) and x_2 is $\underset{\sim}{A}_{12}$(Large) **THEN** y is $\underset{\sim}{B}_1$(Negative)

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$(Large) and x_2 is $\underset{\sim}{A}_{22}$(Small) **THEN** y is $\underset{\sim}{B}_2$(Positive)

Fuzzy AND operator: min; Inference method: max-product

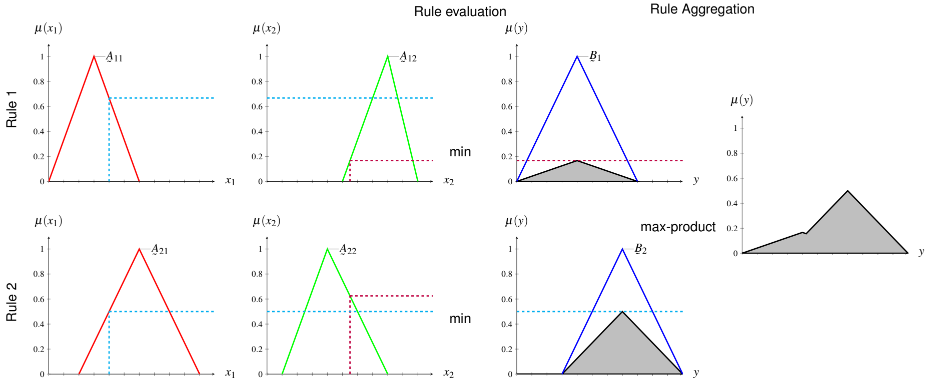

Figure 22: Mamdani (max-product) inference method with crisp inputs. **Grey regions: inferred fuzzy sets**.

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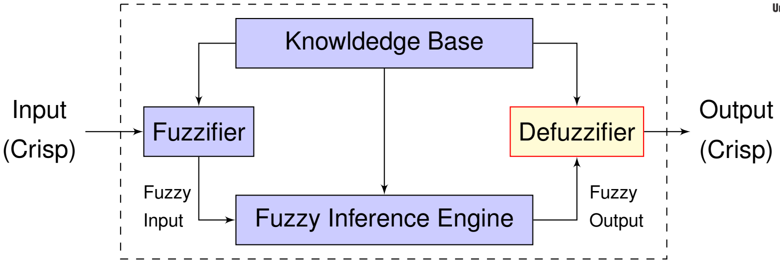
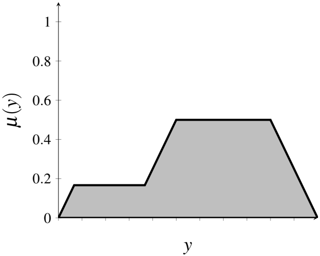
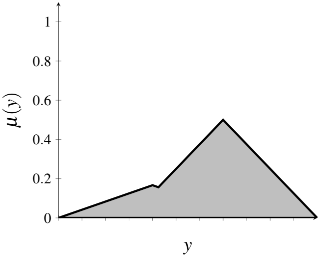


Fuzzy Output: Examples





- Defuzzification is a process to convert the fuzzy output (an inferred membership function) to a crisp value.
- There are a number of methods available for defuzzification, e.g.,
  - *max membership principle*,
  - *centroid method*,
  - *weighted average method*,
  - *mean max membership*,
  - *center of sums*,
  - *center of largest area*,
  - *first (or last) of maxima*.

1. Max Membership Principle

- Also known as the *height method*.
- It is limited to peaked output functions.
- $\mu_{\underset{\sim}{C}}(z^*) \geq \mu_{\underset{\sim}{C}}(z) \forall z \in Z$ where z^* is the defuzzified value.

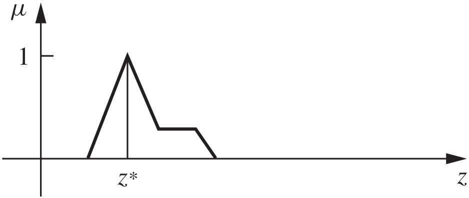


Figure 23: Max membership defuzzification method.

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2. Centroid Method

- Also known as the *center of area* (COA) or *center of gravity* (COG).
- *Continuous form:* $z^* = \frac{\int \mu_{\underset{\sim}{C}}(z) z dz}{\int \mu_{\underset{\sim}{C}}(z) dz}$ where $\int$ denotes an algebraic integration.
- *Discrete form:* $z^* = \frac{\displaystyle\sum_{z_i \in Z} \mu_{\underset{\sim}{C}}(z_i) z_i}{\displaystyle\sum_{z_i \in Z} \mu_{\underset{\sim}{C}}(z_i)}$ where $\sum$ denotes an algebraic sum.

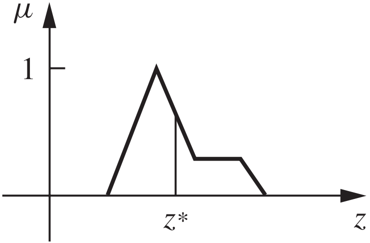

Figure 24: Centroid defuzzification method.

3. Weighted Average Method

- It is computational efficient, however, symmetrical output membership functions are required.
- $z^* = \frac{\sum \mu_{\underset{\sim}{C}}(\bar{z})\bar{z}}{\sum \mu_{\underset{\sim}{C}}(\bar{z})}$ where $\sum$ denotes an algebraic sum and $\bar{z}$ is the centroid of each symmetric inferred membership function.

Example: $z^* = \frac{0.5\times a+0.9\times b}{0.5+0.9}$

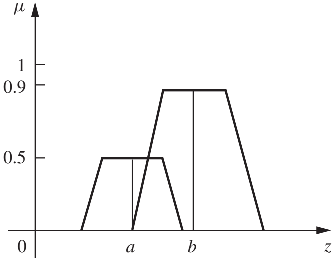

Figure 25: Weighted average defuzzification method.

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4. Mean Max Membership

- Also known as *middle-of-maxima*.
- It is computational efficient.
- $z^* = \frac{a+b}{2}$.

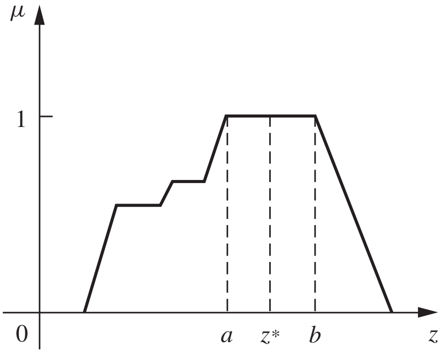

Figure 26: Mean max membership defuzzification method.

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5. Center of Sums

- Faster than many methods. Not restricted to symmetric membership functions.
- This method finds the centroid of the individual output membership functions. The intersecting areas are included twice (*drawback*).
- *Continuous form:* $z^* = \dfrac{\displaystyle\sum_{k=1}^{n} \int \mu_{\underset{\sim}{C}_k}(z)\bar{z}_k dz}{\displaystyle\sum_{k=1}^{n} \int \mu_{\underset{\sim}{C}_k}(z) dz}$ where $\int$ denotes an algebraic integration, $\bar{z}_k$ is the centroid distance of the k^{th} inferred output membership functions.
- *Discrete form:* $z^* = \dfrac{\displaystyle\sum_{k=1}^{n} \sum_{z_i \in Z} \mu_{\underset{\sim}{C}_k}(z_i)\bar{z}_k}{\displaystyle\sum_{k=1}^{n} \sum_{z_i \in Z} \mu_{\underset{\sim}{C}_k}(z_i)}$

  where $\sum$ denotes an algebraic sum.

Example: $z^* = \dfrac{4\times\frac{(4+8)\times 0.5}{2} + 8\times\frac{4\times 1}{2}}{\frac{(4+8)\times 0.5}{2} + \frac{4\times 1}{2}} = 5.6$

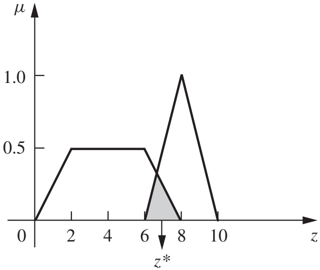

Figure 27: Center of sums defuzzification method.

6. Center of Largest Area

- It is the center of gravity method but the centroid is computed for the largest convex sub-region.
- *Continuous form:* $z^* = \dfrac{\int \mu_{\underset{\sim}{C}_m}(z) z dz}{\int \mu_{\underset{\sim}{C}_m}(z) dz}$ where $\int$ denotes an algebraic integration, $\underset{\sim}{C}_m$ is the largest convex sub-region of the inferred output membership functions.
- *Discrete form:* $z^* = \dfrac{\sum_{z_i \in Z} \mu_{\underset{\sim}{C}_m}(z_i) z_i}{\sum_{z_i \in Z} \mu_{\underset{\sim}{C}_m}(z_i)}$ where $\sum$ denotes an algebraic sum.

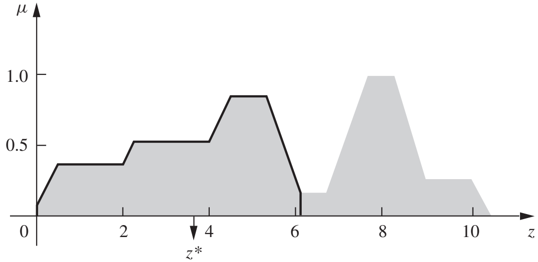

Figure 28: Center of largest area defuzzification method.

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7. First (or last) of Maxima

- *The first of the maxima:* $z^* = \inf_{z \in Z}\{z \in Z | \mu_{\underset{\sim}{C}}(z) = \text{hgt}(\mu_{\underset{\sim}{C}})\}$.
- *The last of maxima:* $z^* = \sup_{z \in Z}\{z \in Z | \mu_{\underset{\sim}{C}}(z) = \text{hgt}(\mu_{\underset{\sim}{C}})\}$

  where *inf* and *sup* stand for *infimum* and *supremum*, respectively.

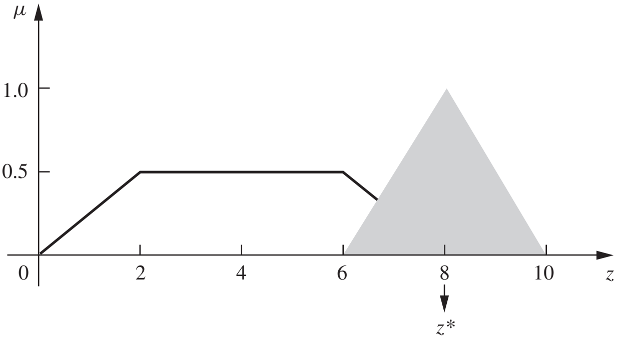

Figure 29: First (or last) of maxima defuzzification method.

Three Fuzzy Inference Systems

Three Fuzzy Inference Systems

- Three common fuzzy inference systems:
  - Mamdani fuzzy inference systems
  - Sugeno fuzzy inference systems (also known as Sugeno fuzzy models, TSK (Takagi, Sugeno, and Kang) fuzzy models)
  - Tsukamoto fuzzy inference systems (also known as Tsukamoto fuzzy models)
- The main difference is in the consequents of the IF-THEN rules
  - Mamdani FIS: Consequent membership function is a general membership function
  - Sugeno FIS: Consequent membership function is a mathematical function
  - Tsukamoto FIS: Consequent membership function is a monotonic membership function (a shoulder function)

1. Mamdani fuzzy inference systems

General rule format:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$ and/or x_2 is $\underset{\sim}{A}_{12}$ and/or $\cdots$ **THEN** y is $\underset{\sim}{B}_1$

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$ and/or x_2 is $\underset{\sim}{A}_{22}$ and/or $\cdots$ **THEN** y is $\underset{\sim}{B}_2$

$\vdots$

Rule r: **IF** x_1 is $\underset{\sim}{A}_{r1}$ and/or x_2 is $\underset{\sim}{A}_{r2}$ and/or $\cdots$ **THEN** y is $\underset{\sim}{B}_r$

- Each consequent is a membership function.
- Rule evaluation and defuzzification are done using any of the introduced methods.

2. Sugeno fuzzy inference systems

General rule format:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$ and/or x_2 is $\underset{\sim}{A}_{12}$ and/or $\cdots$ **THEN** y is $f_1(x_1, x_2, \cdots)$

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$ and/or x_2 is $\underset{\sim}{A}_{22}$ and/or $\cdots$ **THEN** y is $f_2(x_1, x_2, \cdots)$

$\vdots$

Rule r: **IF** x_1 is $\underset{\sim}{A}_{r1}$ and/or x_2 is $\underset{\sim}{A}_{r2}$ and/or $\cdots$ **THEN** y is $f_r(x_1, x_2, \cdots)$

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2. Sugeno fuzzy inference systems

- Each consequent is a function, $f_i(x_1, x_2, \cdots)$, so, each rule has a crisp output.
- When $f_i(x_1, x_2, \cdots)$ is a constant, the Sugeno fuzzy inference system is reduced to Mamdani fuzzy inference system with output membership functions as singletons.
- Rule evaluation is done using any of the introduced methods.
- Defuzzification is obtained by *weighted average* of all functions (Weighted average defuzzification), i.e.,

$$
\begin{aligned}
y &= \frac{w_1(x_1,x_2,\cdots)f_1(x_1,x_2,\cdots)+w_2(x_1,x_2,\cdots)f_2(x_1,x_2,\cdots)+\cdots+w_r(x_1,x_2,\cdots)f_r(x_1,x_2,\cdots)}{w_1(x_1,x_2,\cdots)+w_2(x_1,x_2,\cdots)+\cdots+w_r(x_1,x_2,\cdots)} \\
&= \frac{\sum_{i=1}^{r} w_i(x_1,x_2,\cdots)f_i(x_1,x_2,\cdots)}{\sum_{i=1}^{r} w_i(x_1,x_2,\cdots)}
\end{aligned}
$$

where $w_i(x_1,x_2,\cdots) = \mu_{\underset{\sim}{A}_{i1} \cap \underset{\sim}{A}_{i2} \cap \cdots}(x_1,x_2,\cdots)$, i = 1, 2, ..., r.

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2. Sugeno fuzzy inference systems

Consider a two-rule Sugeno fuzzy model:

Rule 1: **IF** x is $\underset{\sim}{A}_1$ and y is $\underset{\sim}{B}_1$ **THEN** z is $f_1(x,y)$

Rule 2: **IF** x is $\underset{\sim}{A}_2$ and y is $\underset{\sim}{B}_2$ **THEN** z is $f_2(x,y)$

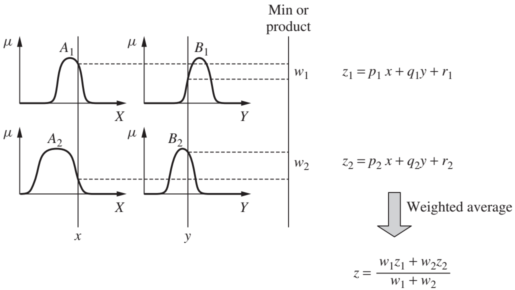

Figure 30: Weighted average defuzzification method for Sugeno fuzzy model.

$w_i(x,y) = \mu_{\underset{\sim}{A}_i \cap \underset{\sim}{B}_i}(x,y)$, $i = 1, 2$.

min: $w_i(x,y) = \min(\mu_{\underset{\sim}{A}_i}(x), \mu_{\underset{\sim}{B}_i}(y))$;   product: $w_i(x,y) = \mu_{\underset{\sim}{A}_i}(x) \times \mu_{\underset{\sim}{B}_i}(y)$

2. Sugeno fuzzy inference systems

Example: An example of 2-input single-output Sugeno fuzzy model with 4 rules:

Rule 1: **IF** x is *Small* and y is *Small* **THEN** z is $-x+y+1$

Rule 2: **IF** x is *Small* and y is *Large* **THEN** z is $-y+3$

Rule 3: **IF** x is *Large* and y is *Small* **THEN** z is $-x+3$

Rule 4: **IF** x is *Large* and y is *Large* **THEN** z is $x+y+2$

$$z = \frac{w_1(-x+y+1)+w_2(-y+3)+w_3(-x+3)+w_4(x+y+2)}{w_1+w_2+w_3+w_4}$$

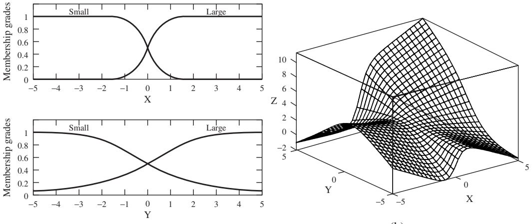

Figure 31: 2-input, single-output Sugeno fuzzy model with 4 rules. (a) Antecedent and consequent membership functions. (b) Overall output surface.

min: $w_1(x,y) = \min(\mu_{x_{Small}}(x), \mu_{y_{Small}}(y))$; $w_2(x,y) = \min(\mu_{x_{Small}}(x), \mu_{y_{Large}}(y))$; $w_3(x,y) = \min(\mu_{x_{Large}}(x), \mu_{y_{Small}}(y))$;

$$w_4(x,y) = \min(\mu_{x_{Large}}(x), \mu_{y_{Large}}(y))$$

product: $w_1(x,y) = \mu_{x_{Small}}(x) \times \mu_{y_{Small}}(y)$; $w_2(x,y) = \mu_{x_{Small}}(x) \times \mu_{y_{Large}}(y)$; $w_3(x,y) = \mu_{x_{Large}}(x) \times \mu_{y_{Small}}(y)$;

$$w_4(x,y) = \mu_{x_{Large}}(x) \times \mu_{y_{Large}}(y)$$

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3. Tsukamoto fuzzy inference systems

General rule format:

Rule 1: **IF** x_1 is $\underset{\sim}{A}_{11}$ and/or x_2 is $\underset{\sim}{A}_{12}$ and/or $\cdots$ **THEN** y is $\underset{\sim}{C}_1$

Rule 2: **IF** x_1 is $\underset{\sim}{A}_{21}$ and/or x_2 is $\underset{\sim}{A}_{22}$ and/or $\cdots$ **THEN** y is $\underset{\sim}{C}_2$

$\vdots$

Rule r: **IF** x_1 is $\underset{\sim}{A}_{r1}$ and/or x_2 is $\underset{\sim}{A}_{r2}$ and/or $\cdots$ **THEN** y is $\underset{\sim}{C}_r$

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3. Tsukamoto fuzzy inference systems

Consider a two-rule Tsukamoto fuzzy model:

Rule 1: **IF** x is $\underset{\sim}{A}_1$ and y is $\underset{\sim}{B}_1$ **THEN** z is $\underset{\sim}{C}_1$

Rule 2: **IF** x is $\underset{\sim}{A}_2$ and y is $\underset{\sim}{B}_2$ **THEN** z is $\underset{\sim}{C}_2$

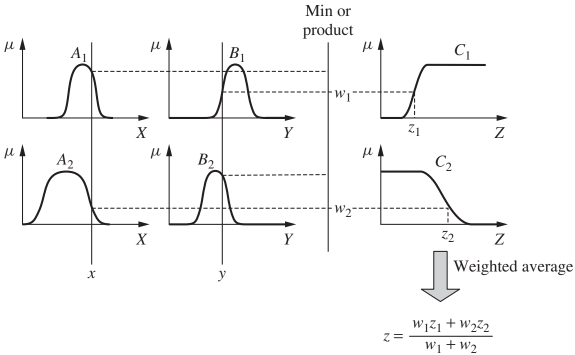

Figure 32: Weighted average defuzzification method for Tsukamoto fuzzy model.

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3. Tsukamoto fuzzy inference systems

Example: An example of single-input single-output Tsukamoto fuzzy model with 3 rules:

Rule 1: **IF** X is $Small$ **THEN** Y is C_1

Rule 2: **IF** X is $Medium$ **THEN** Y is C_2

Rule 3: **IF** X is $Large$ **THEN** Y is C_3

Inferred Output:

$$Y = \frac{\mu_{\text{Small}}(X)C_1(X) + \mu_{\text{Medium}}(X)C_2(X) + \mu_{\text{Large}}(X)C_3(X)}{\mu_{\text{Small}}(X) + \mu_{\text{Medium}}(X) + \mu_{\text{Large}}(X)}$$

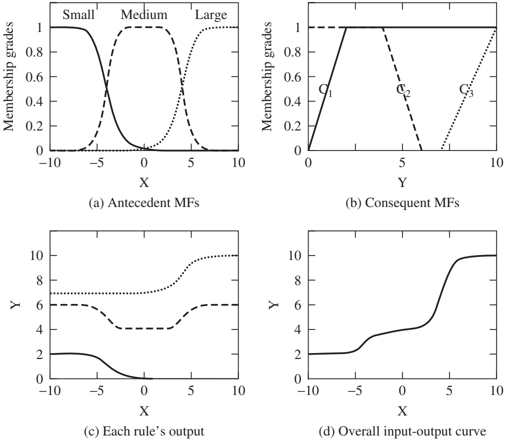

Figure 33: Single-input, single-output Tsukamoto fuzzy model with 3 rules.

Fuzzy Inference System and its Learning

Fuzzy Inference System and its Learning

1. Understand the problem and formulate the problem as an optimisation problem that help define the cost/fitness/objective function
2. Define the FIS, e.g., number of inputs and outputs, number of rules, AND/OR operations, input/output membership functions
3. Define the decisions variables, e.g., the parameters of the membership functions, the coefficients of the functions in the consequents, to be learnt so that to optimise the cost/fitness/objective function
4. Choose a suitable learning algorithm (numerical optimisation, nature-inspired learning algorithms) for learning