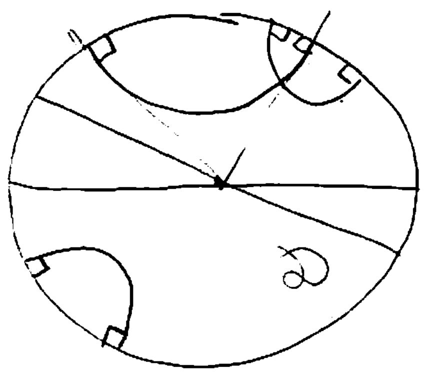
The points in non-euclidean geometry consist of the points in the unit disk

$\mathcal{D} = \{x^2+y^2<1\}$ POINTS STRICTLY INSIDE THE CIRCLE

$\mathcal{C} = \{x^2+y^2=1\}$

The lines (h-line) are the diameters of $\mathcal{D}$ together the ~~part~~ arcs of circles which meet ~~$\mathcal{C}$~~ $\mathcal{C}$ at right angles



What is the equation of an h-line?

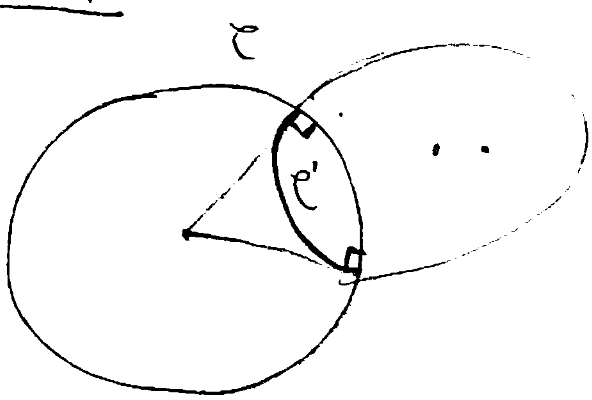
diameters $ax+by=0$

<u>Thm</u> The equation of an h-line is one of the two following forms

1) $ax=by$

2) $x^2+y^2-2ax-2by+1=0$, $a^2+b^2>1$

Proof



The circle C' intersects C in 2 points and at those points the tangent line is the radius

$x^2+y^2-2ax-2ay+c=0$

$(x-a)^2+(y-b)^2-a^2-b^2+c=0$ $\qquad a^2+b^2-c>0$

We want to show that $c=1$.

Let (x_0, y_0) be a point in $C \cap C'$

$$\begin{cases} x_0^2+y_0^2=1 \\ x_0^2+y_0^2-2ax_0-2ay_0+c=0 \end{cases} \Rightarrow \begin{cases} x_0^2+y_0^2=1 \\ -2ax_0-2ay_0+c+1=0 \end{cases}$$

Using the implicit function theorem we have

$2x+2yy'-2a-2ay'=0$ $\qquad$ Where y' is the slope of the line tg to C' at (x,y)

What is the slope of such tg at (x_0, y_0)?

$y' = \dfrac{y_0}{x_0}$

$2x_0+2\dfrac{y_0^2}{x_0}-2a-2a\dfrac{y_0}{x_0}=0 \Rightarrow 2x_0^2+2y_0^2-2ax_0-2ay_0=0$

$$\begin{cases} 2 - 2ax_0 - 2ay_0 = 0 \\ -2ax_0 - 2ay_0 + 1 = 0 \end{cases} \Rightarrow \epsilon = 1$$

<u>Th</u>: Through 2 distinct points in $\mathcal{D}$ there exists a unique h-line. (P2)

<u>Proof</u>

Let $(x_1, y_1), (x_2, y_2)$ be two points in $\mathcal{D}$ and suppose that

$x_1^2 + y_1^2 < 1$, $x_2^2 + y_2^2 < 1$ and suppose that they do not belong to the same line through the origin.

Substituting (x_1, y_1) and (x_2, y_2) in $x^2 + y^2 - 2ax - 2by + 1 = 0$ we obtain 2 eqs.

$$\begin{cases} x_1^2 + y_1^2 - 2ax_1 - 2by_1 + 1 = 0 \\ x_2^2 + y_2^2 - 2ax_2 - 2by_2 + 1 = 0 \end{cases}$$

These 2 eqs determine a <u>unique</u> a and a <u>unique</u> b. This take care of uniqueness.

However in order to prove <u>existence</u> one must check that a and b so obtained satisfy $a^2 + b^2 > 1$.

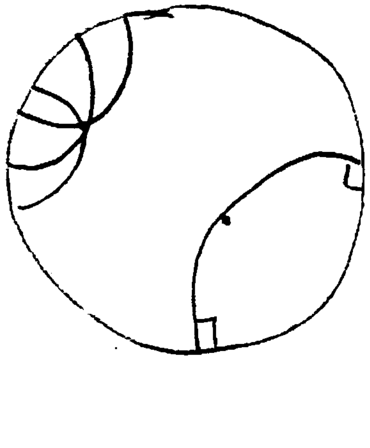
<u>example</u> $(\frac{1}{2}, 0), (0, \frac{1}{2})$

$$\begin{cases} \frac{1}{4} - 2a\frac{1}{2} + 1 = 0 \\ \frac{1}{4} - 2b\frac{1}{2} + 1 = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{5}{4} \\ b = \frac{5}{4} \end{cases}$$

$$x^2 + y^2 - \frac{5}{4}x - \frac{5}{4}y + 1 = 0 \qquad \frac{25}{16} + \frac{25}{16} > 1$$

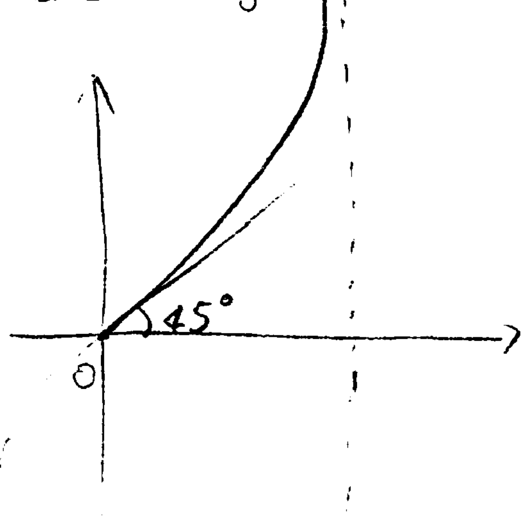
<u>Thm</u>. Given an h-line l and a point p not in l there exist infinitely many h-line through p parallel to (not intersecting) l.

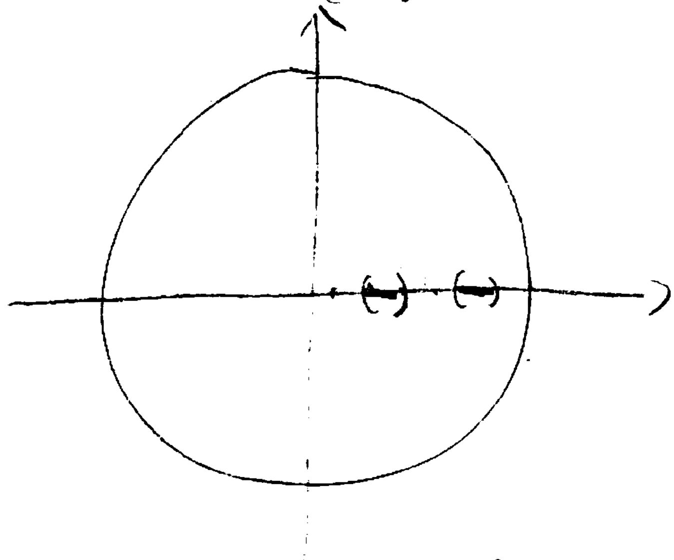
<u>Proof</u> algebra (Parallel postulat does not hold)



Def The h-distance between two points $z_1, z_2 \in \mathbb{D}$ is $d(z_1,z_2) = \tanh^{-1}\left(\left|\frac{z_2 - z_1}{1-\bar{z}_1 z_2}\right|\right)$

Def The ~~chord~~distance from a point z to the origin is $d(0,z) = \tanh^{-1}(|z|)$





Thm Let l be the h-line $\begin{cases} x^2+y^2<1 \\ y=0 \end{cases}$

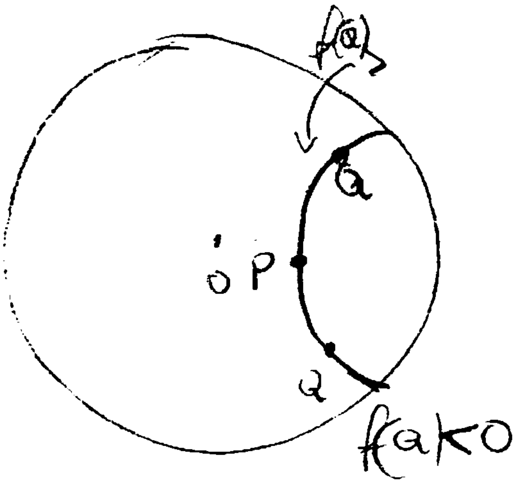
Then the map

$$f: l \longrightarrow \mathbb{R}$$
$$x \longrightarrow \operatorname{sgn} x \ \mathrm{dist}(x, 0) = \operatorname{sgn}$$

is a bijection that respects distances.
A similar bijection can be defined for all h-lines which are diameters. In general, let l be an h-line, not a diameter, and pick a point $p \in l$

$$f: \ell \longrightarrow \mathbb{R}$$

$$a \longmapsto \begin{cases} \text{dist}(P,a) & \text{if to go from } P \text{ to } a \text{ one travels clockwise} \\ -\text{dist}(P,a) & \text{otherwise} \end{cases}$$



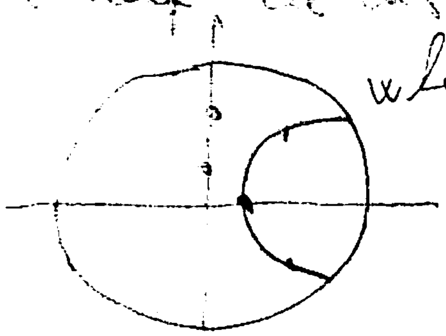
P1

<u>Def.</u> h-segment, h-mid-point, h-half-line, h-triangle, h-circle

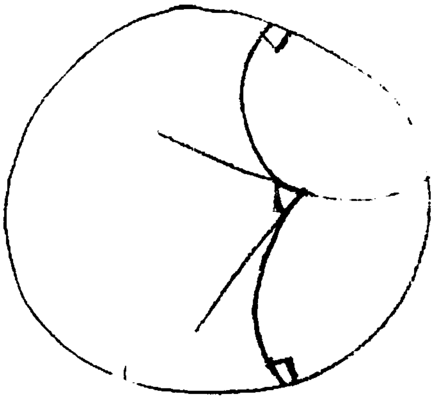
How to find a mid-point.

1) Find the distance between the two points
2) Find the h-segment between the two points
3) Find the point in the h-segment and between the two points which is at half the distance.

where is it going to be?

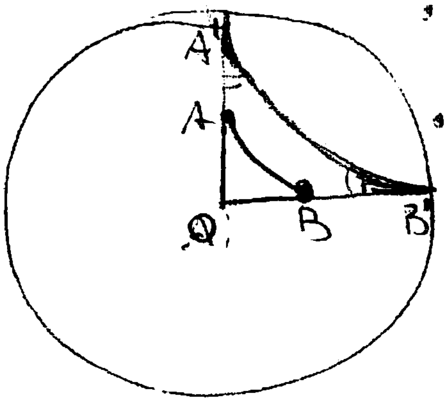


Def The angle between 2 h-half-lines with the same end point is the standard euclidean angle between the tangent half-lines



(P3)

The (SAS) criterion for similarity is not true.



- $OA'B' < OAB$
- $\frac{A'B'}{AB} > \frac{A'O}{AO}$

The sum of the angle of a triangle is $< 180°$.

Th Pythagoras Theorem

Let ABC be an h-triangle with $\angle ACB = 90$ then

$$\cosh 2c = \cosh 2a \cdot \cosh 2b$$

Prop Every λ-circle is an Euclidean circle and viceversa. (The pictures are the same but they mean different things,