Isometries

An isometry of the Euclidean plane is a map that preserves distances

$T: E \to E$ is an isometry if for any $P, Q \in E$ $|T(P) - T(Q)| = |P - Q|$

Examples of isometries.

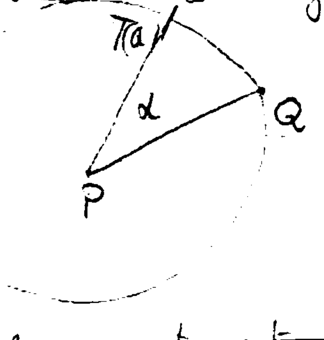
Identity.

Def A rotation of an angle α around a point P is the following map

$T: E \to E$

$T(P) = P$

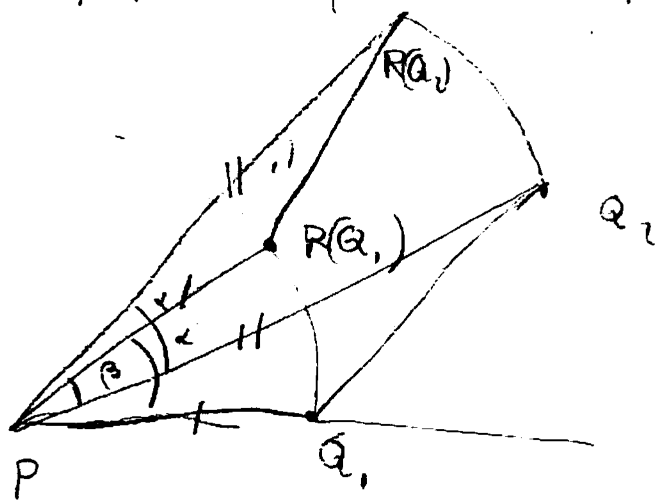
What is $T(Q)$? $T(Q)$ is the point at distance $|P-Q|$ from P that lies on the half-line making an angle α with PQ



$T(Q)$ is the point ~~at the~~ obtained by intersecting the circle centered at P of radius $|P-Q|$ with the half-line that makes an angle α with P,Q.

Prop^n A rotation is an isometry.

Proof Let $R_{P,\alpha}$ be a rotation ~~of~~ to angle α around P. We want to show that for any $Q_1, Q_2 \in E$ $|Q_1 - Q_2| = |R(Q_1) - R(Q_2)|$



$\angle R(Q_2) P R(Q_1) = \alpha - \beta$

$\angle Q_2 P Q_1 = \alpha - \beta$

Case 1 of similarity.

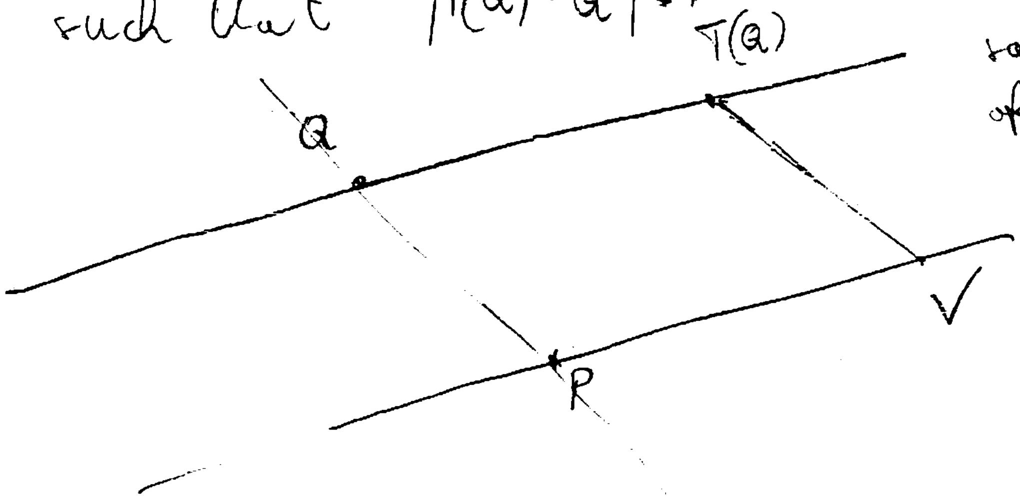
Def A traslation of PV is the following map

$T : E \to E$.

$T(P) = V$

For any $Q \in E$, what is $T(Q)$?

Consider the unique line parallel to PV containing Q, l_Q. Then $T(Q)$ is the unique point on the l_Q such that $|T(Q) - Q| = PV$ and on the same side of PQ as V.

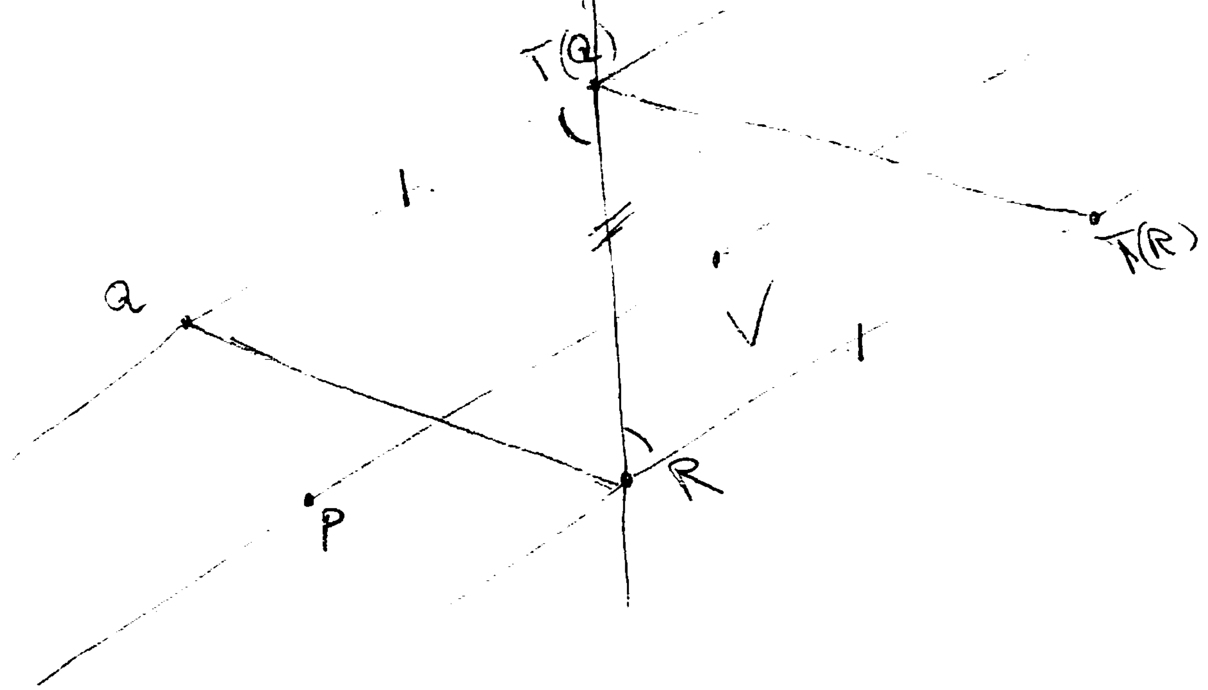


Precisely the line PQ separates the Euclidean plane into two half-planes. To be on the same side means ..

Prop A translation is an isometry

Proof. Let T be a translation of PV

We want to show that $|T(Q) - T(R)| = |Q - R|$



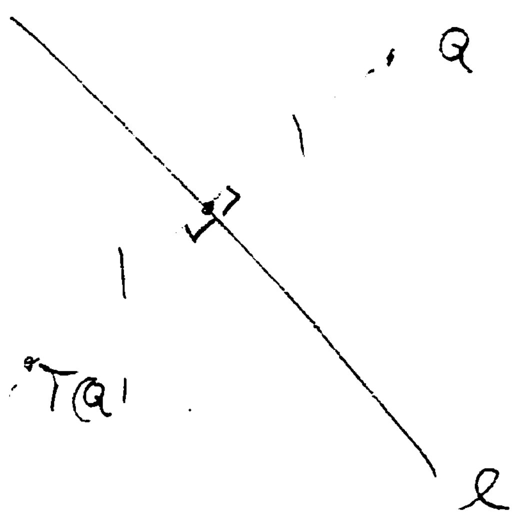
Def Reflection across a line l is the following map

$T : E \to E$.

$T(P) = P$ for any $P \in l$

if $Q \notin l$ then let m_Q be the perpendicular to l and let $R = l \cap m_Q$. $T(Q)$ is the unique point ~~on~~ m_Q different from Q s.t.

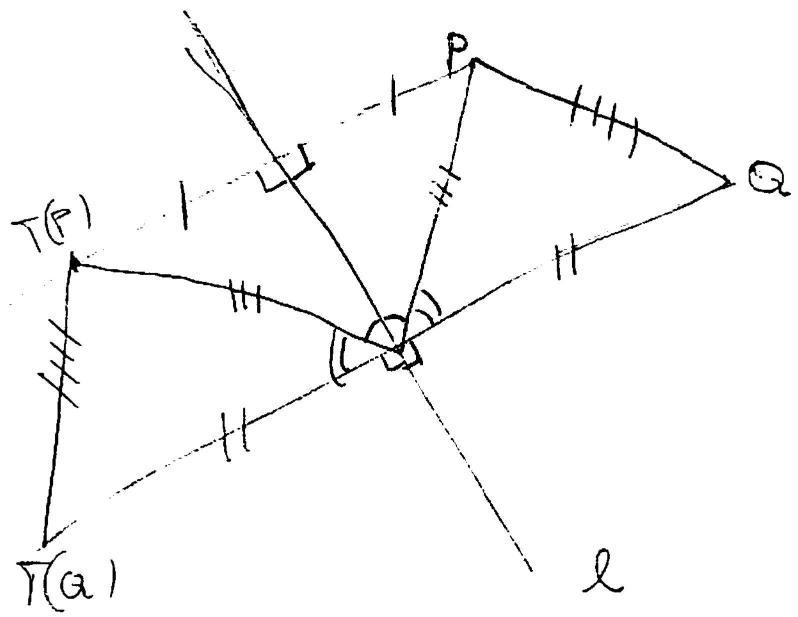
$|T(Q) - R| = |Q - R|$



<u>Prop</u> A reflection is an isometry

<u>Proof</u>

$$|T(P) - T(Q)| = |P - Q|$$



Th The composition of two isometries is an isometry and the inverse of an isometry is an isometry.

Proof

Let T_1 and T_2 be two isometries.

We want to show that for any $P, Q \in E$

$$|T_1 \circ T_2(P) - T_1 \circ T_2(Q)| = |P - Q|$$

$$\parallel \qquad\qquad //$$

$$|T_2(P) - T_2(Q)|$$

Let T be an isometry.

$$|T^{-1}(Q) - T^{-1}(P)| = |P - Q|$$

$R = T^{-1}(Q)$ if $T(R) = Q$ and $S = T^{-1}(P)$ if $T(S) = P$

$$|P - Q| = |T(R) - T(S)|$$

$$\parallel$$

$$|T^{-1}(Q) - T^{-1}(P)| = |T^{-1}T(R) - T^{-1}T(S)| = |R - S|$$

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Th The set of isometries consists of identity, rotations, translations, line reflections, their inverses and their compositions.

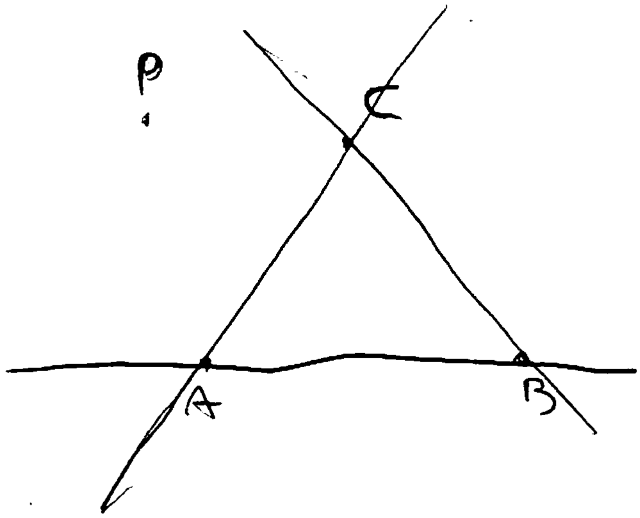
Th An isometry that fixes two points A, B fixes all the points on the line containing AB. Moreover, it is either a reflection across such line or, if it fixes a point C not on the line, it is the identity.

Proof Let T be such isometry and let l be the line through A and B and let P be a point on l.
Consider the circle C_{PA} centered at A of radius PA and the circle C_{PB} centered at B of radius PB. $P \in C_{PA} \cap C_{PB}$.

$$|T(P) - T(A)| = |P - A| \qquad |T(P) - T(B)| = |P - B|$$
$$|T(P) - A| \qquad\qquad |T(P) - B|$$

Thus $T(P) \in C_{PA} \cap C_{PB}$. If $T(P) \neq P$ then ~~AB lies on~~ is the perpendicular bisector to $P\,T(P)$. However, since $P \in l$ this is impossible.

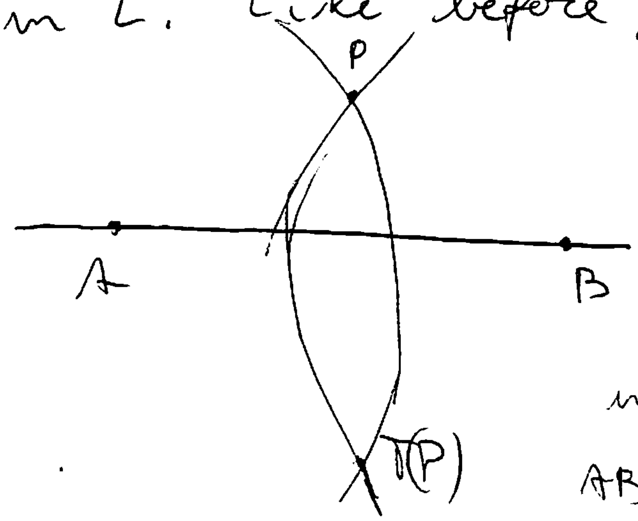
~~Similarly,~~ Suppose $C \notin l$ and $T(C) = C$.



The ~~two~~ lines ~~through~~ AC and CB are lines of fixed points

Let P be a general point and let m_P be any line through P. Such line intersects at least two of the lines (it can't be parallel to all three) thus it has two fixed points and P is also a fixed point.

Suppose T does not have any other fixed point. Let P be a point in E not in L. Like before, consider the circles centered at A and B of radius AP and BP.



Exercise: T(P) is the other point of intersection. Since AB is the perpendicular bisector to P T(P), T(P) is the points symmetric to P across the line AB.

Th An isometry is a rotation if and only if it fixes exactly one point.

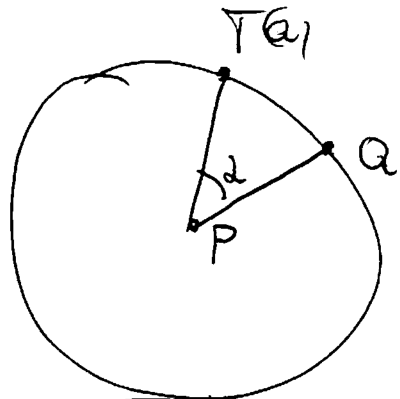
Proof

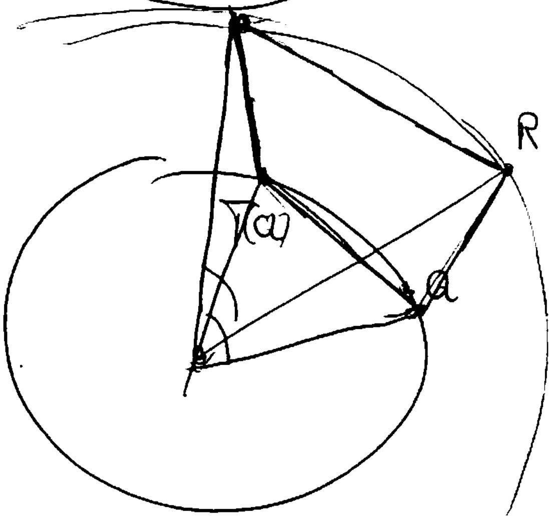
Suppose P is the only point such that $T(P)=P$.

Let Q be a point in E different from P.

Then $T(Q)$ is on the circle centered at P of radius PQ and $PT(Q)$ makes an angle $\alpha \in (0, 180^\circ)$ with $T(Q)$,

Let R be another point. We need to show that $PT(R)$ makes an angle α with PR.





Thm All isometries

Proof.

~~Suppose~~

If the isometry has at least a fixed point then we are done.

Suppose not. Let $T(P)=Q$ and consider the new isometry $G=t\circ T$ where t is a translation taking Q to P.

Then $G(P)=P$ and it is known

$$G=t\circ T \quad \text{thus} \quad T=t^{-1}\circ G.$$