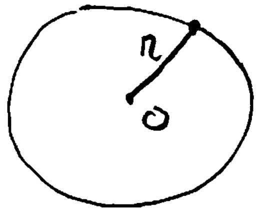
4 Circles, chords and arcs.

Def All points at a given distance from a given fixed point are said to form a circle. The given fixed point is the center of the circle and the given distance is the radius.

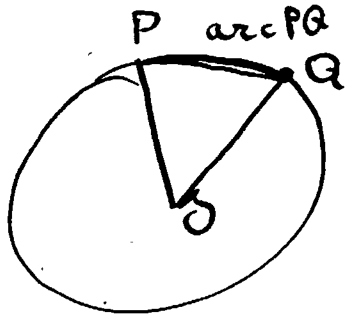


Consider a circle centered at o of radius r.
Every point at distance

$$\left\{\begin{array}{l}\text{equal to } r\\ \text{less than } r\\ \text{greater than } r\end{array}\right\} \text{ from } O \text{ is said to be } \left\{\begin{array}{l}\text{on the circle}\\ \text{inside the circle}\\ \text{outside the circle}\end{array}\right\}$$

~~Consider circle centered at o of radius r on and a half-line ox starting at o.~~
~~We may select ox as a reference line from which to meo~~

Given two points P and Q on a circle. All the points ~~of the circle~~ they which lie on the half-lines between P and Q are said to for an arc PQ of the circle



The angle POQ is called the central angle.

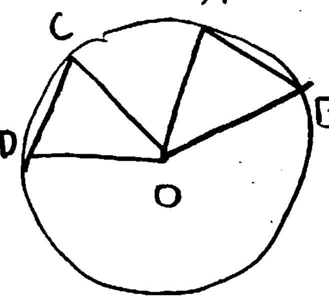
Recall we usully refer to the lesser of the two angles and thus to the "lesser" of the two chords. If the two central angles are equal then arc PQ is a semi-circle.
We say that two arcs are equal if their central angles are equal.

A straight line segment forming two point of a circle is called a chord.

A chord which passes through the center is called a diameter.
Abusing the notation we will use radius and diameter to denote both distances an line segments.

Thm Equal chords have equal arcs.

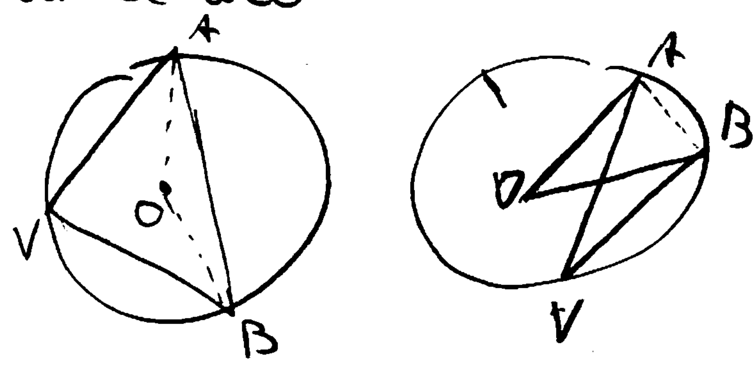
Proof Suppose $AB = CD$ we want to show that arc AB = arc CD or equivalently, $\angle AOB = \angle DOC$.



However since $AB = CD$ and $OD = OC = OA = OB$ AOB and COD are similar by SSS (In fact equal). Thus $\angle AOB = \angle COD$. Viceversa uses SAS!

Defn An angle whose vertex is on the circle and sides are chords is a inscribed angle.

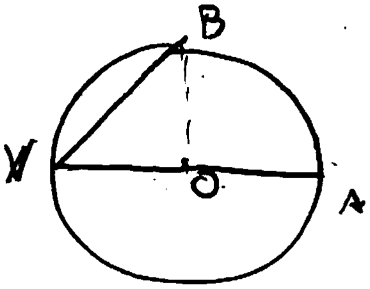
Theorem An inscribed angle is equal to half the central angle having the same arc



Proof We want to show that $\angle AVB = \frac{1}{2} \angle AOB$.

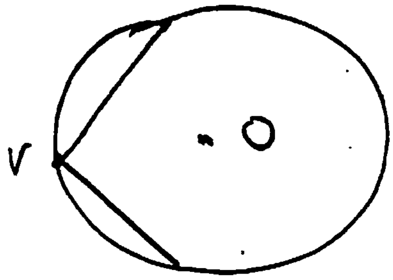
Three cases

Case 1



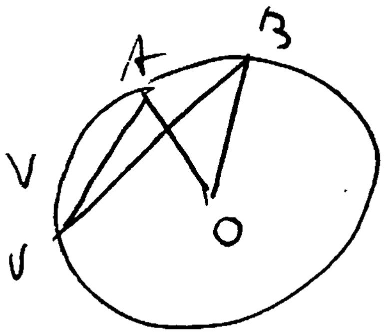
O lies on one side of $\angle AVB$.

Case 2



O lies inside $\angle AVB$

Case 3



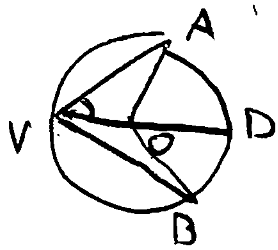
O lies outside $\angle AVB$

Proof

Case 1 $\angle AOB = 180^\circ - \angle VOB = \cancel{180^\circ}\ \angle VOB + \angle VBO =$

$= 2\angle VOB$.

Case 2 Draw the diameter from V.



It divides $\angle AVB$ in 2 parts, each of which falls into case 1. Thus $\angle AVO = \frac{1}{2}\angle AOD$ and $\angle OVB = \frac{1}{2}\angle DOV$ thus it follows

Case 3

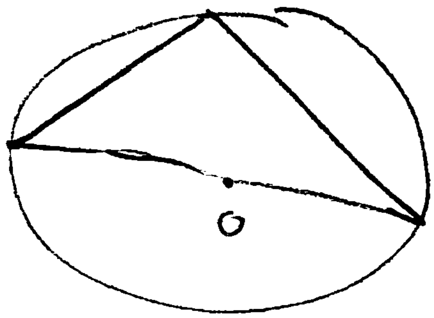
Again ~~draw~~ draw the diameter and apply case 1.

<u>Corollary</u> Equal angles inscribed in the same circle have equal arcs.

<u>Cor</u> All inscribed angles having the same arc are equal.

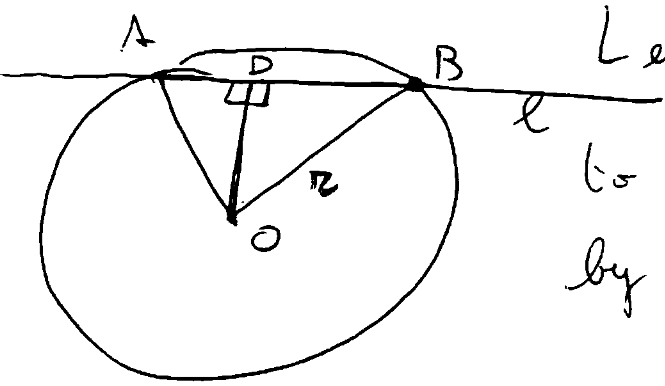
<u>Cor</u> Every angle inscribed in a semi-circle is a right angle

<u>Cor</u> Thales Theorem. A triangle inscribed in a circle whose a side is a diameter is a right triangle



Prop. A line and a circle have at most 2 points in common.

Proof. Let l be such line and let A, B be two distinct ~~of the~~ points of intersection.



Let OD be the perpendicular to AB, then $AD = DB$. by the Pythagorean Theorem.

Suppose there exists another point C on l such that $OC = r$ then, just like before,

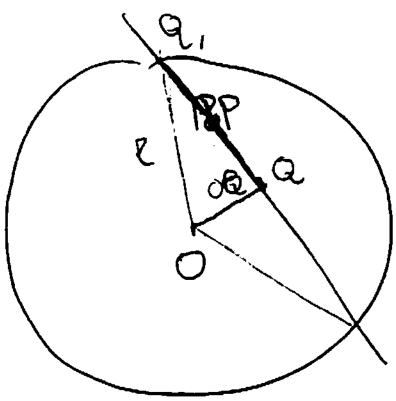
$DC = AD = DB$. However, we have proven that given a point P on a line and a distance d, there are ~~exactly~~ to points on such line at distance d from P. Thus C is either A or B.

Def A line intersecting a circle in 2 points is called secant.
A line intersecting a circle in 1 point is called tangent.

Theorem: If a line ℓ has a point inside a circle $\mathcal{C}_r$ then it is a secant line.

Proof. Let O be the center of $\mathcal{C}_r$.
Let P be the point on ℓ inside $\mathcal{C}_r$ and let Q be the point on ℓ s.t. OQ is perpendicular to ℓ at Q.

Since Q is the point on ℓ that realizes the least distance from O to ℓ,
we have that $OQ \leqslant OP < r$



Let Q_1 and Q_2 be the 2 points at distance $\sqrt{r^2 - OQ^2}$ from Q then

OQ_1Q is a right triangle and

$$OQ_1 = \sqrt{OQ^2 + Q_1Q^2} = r$$

Thus Q_1 is on the circle. Same for Q_2.

Theorem Let l be a line intersecting a circle $\mathcal{C}_r$ at a point P.
l is tangent to $\mathcal{C}_r$ at P if and only if OP is perpendicular to l.

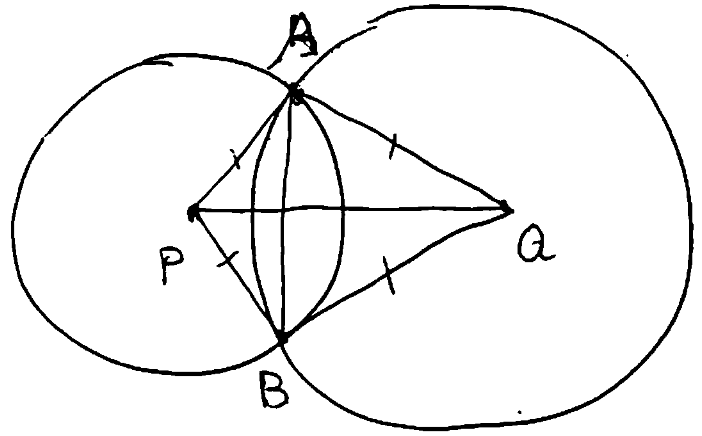
Proof

Part 1 l being tangent to $\mathcal{C}_r$ at P means that P is the only point of intersection.
If OP is not perpendicular to l then there exists a point in l which is closer to O than P is. ~~Because~~ Since P is on $\mathcal{C}_r$, being closer to O implies being inside $\mathcal{C}_r$. However, if that is the case then l would intersect $\mathcal{C}_r$ at 2 points (previous theorem).

Part 2: If OP is perpendicular to l then for any point $Q \in l$, $Q \neq P$, $OQ > OP = r$
Thus Q cannot be on the circle.

Thm If two circles C_r, C_s (centers P, Q) intersect at exactly two points ~~then~~ A, B then PQ lies on the perpendicular bysector of AB.

Proof



P and Q are equidistant from A and B.

Cor. Two circles have at most 2 points in common.

Proof. Let A, B, C be three point in common. then PQ is perpendicular to AB and BC. This implies that ~~$\angle ABC$~~ $AB \parallel BC$ which implies that $\angle ABC = 180°$. But a line only intersect a circle in two points so A, B, C cannot be distinct.

Thm If A is the unique point o intersection the A lies on PQ.