TRIANGLES

Theorem 1 (Case 2 of similarity) (AAA)
Two triangles are similar if two angles of one are equal to two angles of the other.
Let ABC, A'B'C' be 2 triangles such that $\angle A = \angle A'$ and $\angle B = \angle B'$ then $\angle C = \angle C'$ and

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

Q: What is given?

A: Two triangles ABC and A'B'C' with the property that $\angle A = \angle A'$ and $\angle B = \angle B'$

Q: What do we want to prove?

A: $\angle C = \angle C'$ and $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'} = \dfrac{BC}{B'C'}$

Q: How can we prove that 2 triangles are similar?

A: Using Postulate 4, i.e. Case 1 of similarity?

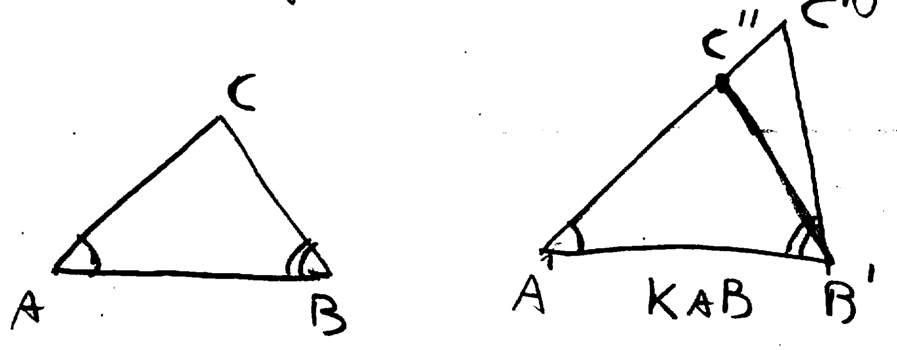
Q: Therefore, what do we need to show?

A: To apply Postulate 4 we need to show that

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} \quad \left(\text{or } \frac{BA}{B'A'} = \frac{BC}{B'C'}\right)$$

<u>Proof</u> We cannot apply Postulate 4 directly.

We are going to use it indirectly.



Let k s.t. $A'B' = KAB$. Let us construct a triangle $A'B'C''$ similar to ABC and then ~~show~~ that $A'B'C' = A'B'C''$.

Let C'' ~~be~~ a point on the ~~ray~~ $A'C'$ s.t. (P1) $A'C'' = KAC$. By (P4) $ABC \simeq A'B'C''$

We would like to say that $C' = C''$.

Note C' is the intersection between $A'C'$ and $B'C'$.

C'' lies on $A'C'$. If we can show that it also lies ~~on~~ $B'C'$ then, since 2 lines have (P2) at most 1 point in common $C' = C''$.

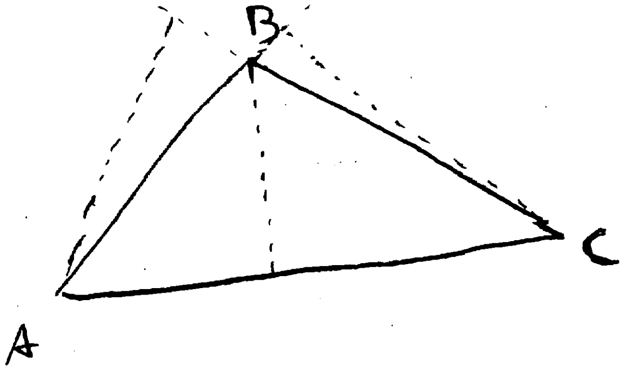
We know that $ABC = A'B'C'$ by hypothesis and

$ABC = A'B'C''$ by similarity

then $A'B'C' = A'B'C''$ which shows that (P3) C'' is on the line $B'C'$.

Why it is not the other ray? Because the angle is $<180°$.

Def The line segment from the vertex of a triangle to the line determined by the segment opposite to such vertex is called an altitude of the triangle (by altitude we also mean the length of such segment).

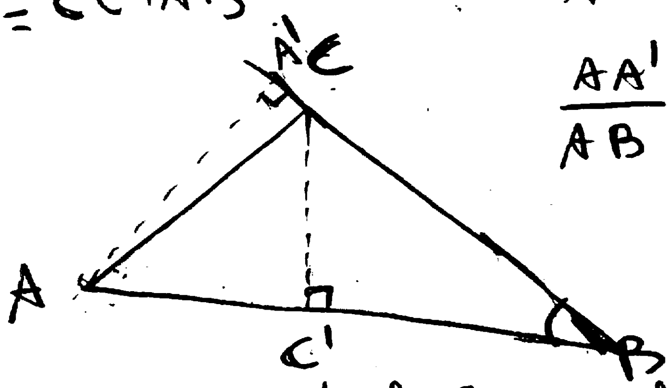


Prop. The product of an altitude and the corresponding side of a given triangle is constant for a given triangle

Proof Given a triangle ABC let A', B', C' be respectively the foot of the altitude starting at A, B, C. We want to show that $AA' \cdot BC = BB' \cdot AC = CC' \cdot AB$

$$AA' CB = CC' AB$$

$$\frac{AA'}{AB} = \frac{CC'}{CB}$$



This would follow if we can show that $AA'B$ and $CC'B$ are similar triangles.

(P5) won't work but Theorem 1 does.

$\sphericalangle ABA' = \sphericalangle CBC'$ and $\sphericalangle AA'C = 90° = \sphericalangle CC'B$

Thus, by theorem 1, having two equal angles, ABA' and CBC' are similar triangles.

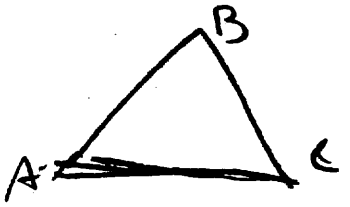
<u>Thm 2</u> If 2 sides of a triangle are equal then the angles opposite to these sides are equal. and viceversa.

Pons asinorum or bridge of asses.

One explanation; in Euclid's proof the diagram looks like a bridge

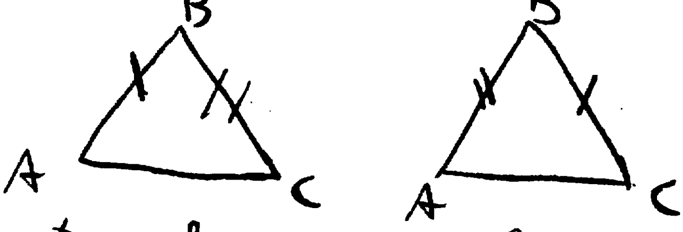
More popular one: it is the first real test in geometry, the first bridge to cross and if you can't ...

<u>Proof</u> Given a triangle ABC with $AB = BC$ we want to show that $\sphericalangle A = \sphericalangle C$



So far, we have made no difference between the triangle ABC and CBA.

In this case we do.



triangle 1   2

$AB = BC$ (P4)

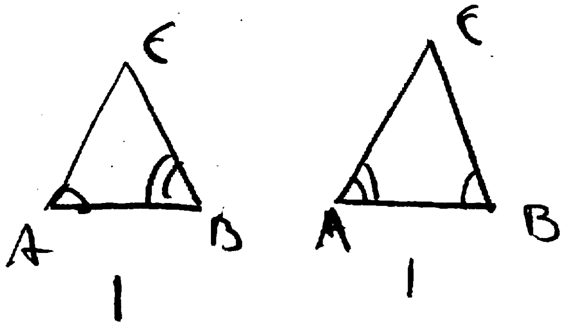
$BC = AB \Rightarrow$ ABC and CBA are similar

$\sphericalangle B = \sphericalangle B$

but in this new way of ordering points.
Thus, the angle opposite to AB in triangle 1
is equal to that opposite BC in 2.

$\sphericalangle BCA = \sphericalangle BAC$.

Viceversa,



ACB and BCA are similar

$AC = K\,CB$
$CB = K\,AC$
$AB = K\,AB \Rightarrow K = 1$ and $AC = CB$

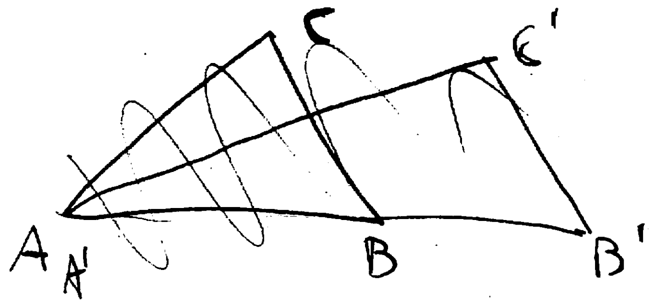
Def: A triangle with two sides equal is called an isosceles triangle (isosceles means equal legs).

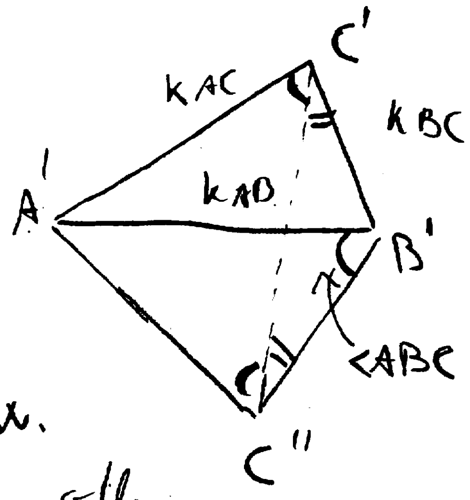
<u>Exercise</u>: If all three sides of a triangle are equal then all three angles are equal and viceversa

<u>Def</u> A triangle in which all sides are equal is called an <u>equilateral</u> triangle.

Theorem 3 Case 3 of similarity (SSS)
Two triangles are similar if their sides are proportional.

Proof Let ABC and $A'B'C'$ be 2 triangles such that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} = k$. Then we want to show that the angles are equal.
By (Pa5) it suffices to show that two angles, say $\sphericalangle ACB$ and $\sphericalangle A'C'B'$ are equal.





Construct $A'B'C''$ as follows.
Let C'' be the point on the other side of $A'B'$ s.t. $\sphericalangle A'B'C'' = \sphericalangle ABC$ and $B'C'' = kBC$

By construction $\frac{A'B'}{AB} = \frac{B'C'}{BC}$ and $\sphericalangle A'B'C'' = \sphericalangle ABC$
and by assumption $A'B' = kAB$
thus (P5) they are similar ~~and~~ $A'C'' = kAC$ and
$\sphericalangle A'C''B' = \sphericalangle ACB$

Since by assumption $A'C' = KAC$

The triangle $A'C'C''$ is isosceles and

thus $\sphericalangle A'C'C'' = A'C''C'$. (Theorem 2)

Similarly, $C'C''B' = C''C'B'$.

Thus $\sphericalangle A'C''B' = \sphericalangle A'C'B'$

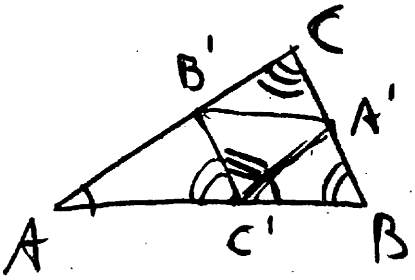
$\|$ $\Leftarrow$ because $A'C''B' \simeq ACB$

$\sphericalangle ACB$

Finally $\sphericalangle ACB = \sphericalangle A'C'B'$ ⩕

<u>Theorem 4</u> The sum of the angles of a triangle is $180°$

<u>Proof</u>



Let A', B', C' be the midpoints of respectively CB, AC, AB

We want to show that

$\sphericalangle CAB = \sphericalangle A'C'B$, $\sphericalangle ABC = \sphericalangle AC'B'$, $\sphericalangle ACB = \sphericalangle A'C'B'$

Let's first show that $ABC \simeq A'BC'$.

This follow from P5 because $A'B = \frac{1}{2} BC$, $C'B = \frac{1}{2} AB$

and $\sphericalangle ABC = C'BA'$.

Similarly $ABC \simeq AB'C'$ which gives $\sphericalangle AC'B' = \sphericalangle ABC$.

We are left to show that $A'C'B' \cong A'B'C$.

They have one side in common $A'B'$

Why is $B'C' = A'C$? $A'C = \frac{1}{2}AB$ by construction

$B'C' = \frac{1}{2}BC$ by similarity of ABC and $AB'C'$.

So $A'C = B'C'$. Similarly $B'C = A'C'$.

$\#$

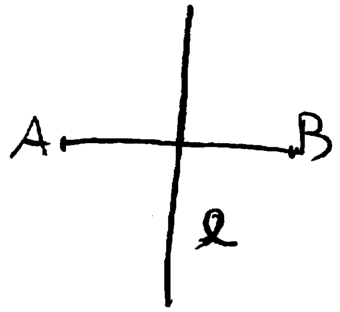
<u>Def</u> . A triangle all of whose angles are acute is called an acute triangle

. A triangle one of whose angles is a right angle is called a right triangle

. A triangle one of whose angle is an obtuse angle is called an obtuse triangle.

<u>Note</u> by Th 4 only one angle can be obtuse.

<u>Thm</u> 5 All points equidistant from the endpoints of a line segment and no other lie on the perpendicular bisector of the line segment.



Let AB be a line segment and let l be its unique perpendicular bisector then

a point P is in l iff $AP = PB$

Proof

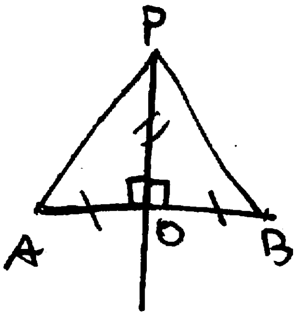
False idea: let P be a point such that

AP = PB and let ~~sk~~' be the perpendicular
then because of the uniqueness of the perpend
from P to AB. We must show that l' = l.

Wrong Why must l' exist?

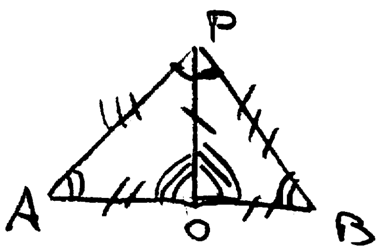
Real proof.

Gouten $P \in l$, let's show that $AP = PB$



The triangle AOP and BOP are equal.

Viceversa, suppose $AP = BP$ and consider the line segment PO. Then $AOP = BOP$.



$\angle AOP + \angle OPA + \angle PAO = 180°$

$\angle AOP + \angle POB = 180°$

$\angle PAO + \angle ABP + \angle BPA = 180°$ and $\angle AOP = \angle POB$

$\angle AOP + \angle OPA + \angle PAO$

$\angle AOP + \angle OPA + \angle PAO = 180° - \angle PAO + \angle ABP - \angle BPA = 0$

$\angle AOP - \angle PAO - \angle AOP = 0$

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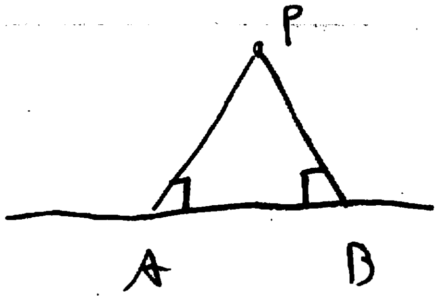
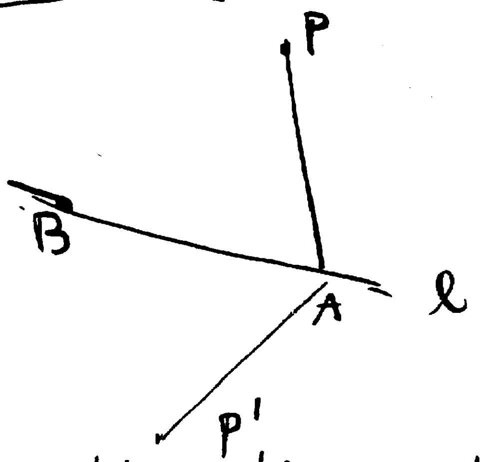
Theorem 6 Through a point not on a line there is exactly one perpendicular

Proof First why can't there be 2 distinct perpendiculars



Sum of the angles of ABP > 180°. Not possible.

Existence Let's construct this perpendicular



Pick a point A on l. If PA is perpendicular to l we are done if not, construct a point P'

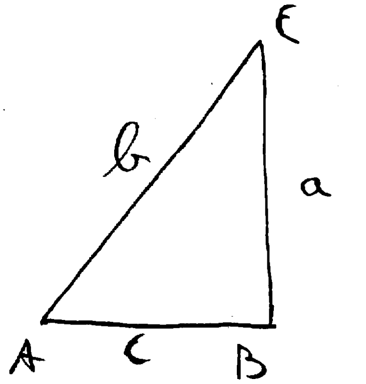
on the other side of l s.t. $\angle PAB = \angle P'AB$

and $AP' = AP$. By construction A is equidistant from P and P' which, by theorem 5, implies that l is perpendicular to PP' or viceversa, that PP' is perpendicular to l.

Def. In a right triangle the side opposite to the right angle is called hypothenuse (also the length). The other sides are called catheti (cathetus) or legs.

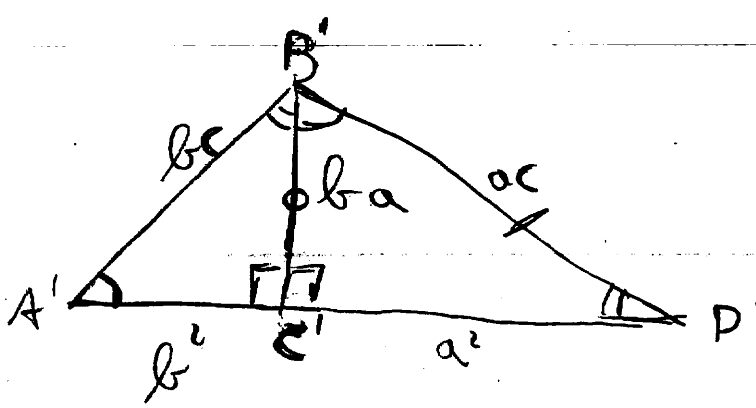
Thm 7 (Pythagorean Theorem)

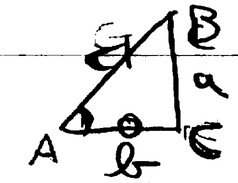
In any right triangle the square of the hypothenuse is equal to the sum of the squares of the other two sides. Viceversa, If in a triangle, the square of one side is equal to the sum of the squares of the other two sides, such triangle is a right triangle



$a^2+c^2=b^2$

<u>Proof</u> Scale the triangle ABC by a factor b





and construct the triangle $B'D'C'$

s.t. $\angle C'B'D' = \angle B'A'C' = \angle BAC$ and $B'D' = ac$

Then, from (P5) $B'C'D' \simeq ABC$ with factor a

Thus $C'D = a^2$

$\angle A'B'D' = \angle A'B'C' + \angle C'B'D' =$

$= 180° - 90° - \cancel{\angle B'A'C'} + \cancel{\angle C'B'D'} = 90° = \angle ACB$

Therefore, the triangles ABC and $A'B'D'$ have 2 equal angles $\angle A'B'D' = \angle ACB$ and $\angle B'A'C' = \angle B'A'D'$

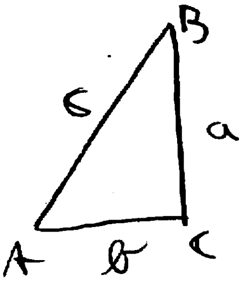
(Note $\angle A'C'B' = 90° = \angle B'C'D'$, thus $\angle A'C'D' = 180°$ meaning $A'C'D'$ are colinear)

The scaling factor is c because $A'B' = bc \simeq AC = b$

Therefore $\underset{\text{hypothenuse of } A'B'D'}{A'B'} = a^2 + b^2 = cAB = c^2$

Viceversa

Let ABC be a triangle s.t.

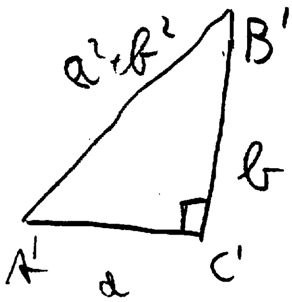


$a^2+b^2=c^2$

We want to show that ABC is a right triangle. We are going to show that it is similar to a right triangle

Construct a right triangle

$A'B'C'$ s.t $A'C'=b$ $B'C'=a$ and $\angle A'C'B'=90^0$



By the previous part of the theorem, $A'B'=a^2+b^2$.

Thus, $ABC \simeq A'B'C'$ (in fact equal)

Therefore $\angle ACB = \angle A'C'B' = 90^\circ$

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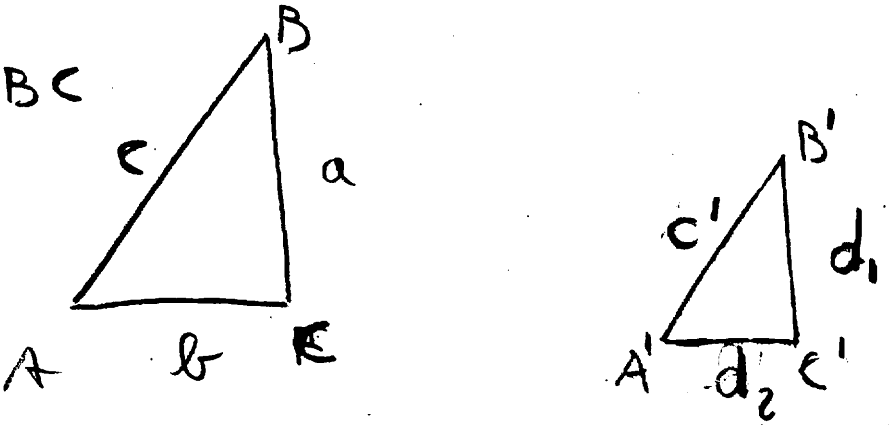
<u>Coriollary (7b)</u> If the hypotenuse and another side of two right triangles are in proportion the the two right triangle are similar.

<u>Proof</u> Two sides in proportion + an equal right angle which is NOT the one in between

ABC



$c' = kc$ and $d_1 = ka$

Since they are right triangle, given 2 sides we can compute the third.
If they are similar then they will be proportional.

$$d_2^2 = (c')^2 - (d_1)^2 = k^2(c^2 - a^2) = k^2 b^2$$

Pythagorean thm

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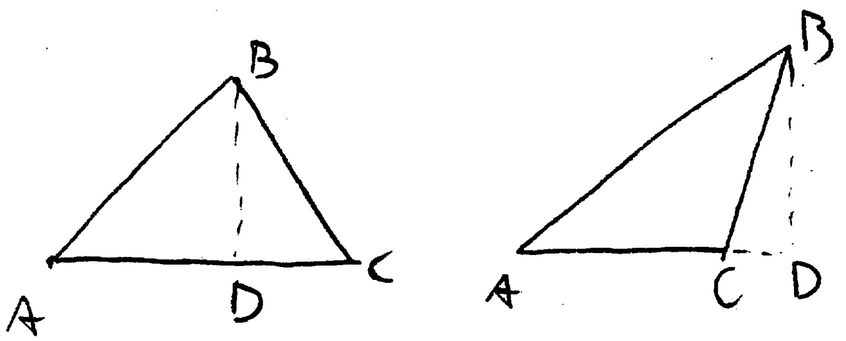
Corollary (7.c) Two right triangles are congruent iff the hypothenuse and a leg of one are equal to the hypothenus and a leg of the other

Corollary (7d) The sum of two sides of a triangle is greater than the third side

<u>Proof</u> Let ABC be any triangle

We have to state that $AB+BC > AC$

Let D be the point on the line AC where the altitude at B intersect



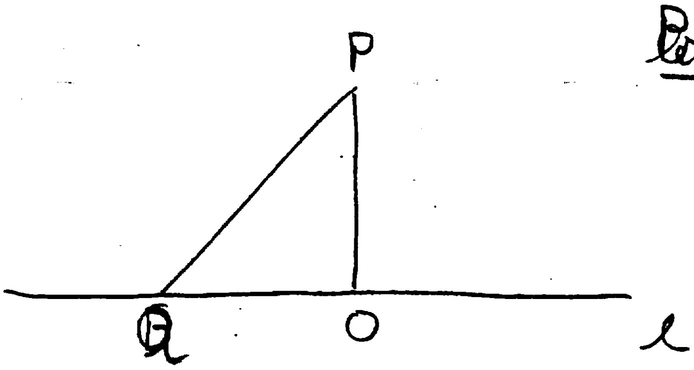
By the Pythagorean theorem $AB^2 = AD^2 + BD^2$

therefore $(AB)^2 > (AD)^2 \implies AB > AD$

Similarly $BC > DC$

$$AB+BC > AD+DC \geqslant AC$$

<u>Corollary 7a</u> The shortest segment from a point to a line is measured along the perpendicular from the point to the line



<u>Proof</u> Let OP be the segment from P perpendicular to l and let PQ be any other segment

We want to show that $PQ > PO$

By the Pythagorean theorem

$$(PQ)^2 = (PO)^2 + (OQ)^2 \Rightarrow PQ > PO.$$

<u>Notation</u> When we speak of distance from point to a line we mean "perpendicular distance," i.e. the length of the perpendicular segment

The point O is called the <u>foot</u> of the perpendicular from P to l.