EUCLIDEAN GEOMETRY: THE AXIOMS.

We will use Birkhoff's set of axioms for Euclidean geo

It consists of 4 postulates but relies on the postulates of real number.

These are the rules of the game

<u>Undefined terms</u>: number, order, equal, point, (straight) line, distance between 2 points, angle between 2 lines.

It helps to think of points and lines as the usual but in fact this is just a model for Euclidean geometry.

Postulate 1 (PRINCIPLE OF LINE MEASURE)

There is a bijection between the points on a straight line and the real numbers and this bijection respect distances

In other words, there exists a map

$f : \{\text{points on a line}\} \longrightarrow \mathbb{R}$. ($\{\text{points on a line}\} = \ell$)

- for any ~~pt~~ $x \in \mathbb{R}$ there exists a $P \in \ell$ such that $f(P) = x$ (surjective)
- if $P, Q \in \ell$ and $P \neq Q$ then $f(P) \neq f(Q)$ (injective).

This map f can be used to measure the distance between two points.

- For any $P, Q \in \ell$ distance from P to Q $d(P, Q)$ is equal to $|f(P) - f(Q)|$.

Note: we are not defining distances we are ~~measu~~ just measuring distances.

In other words, if $d(P, Q) = d(P, R)$ then $|f(P) - f(Q)| = |f(P) - f(R)|$

- We are putting a ruler on a line.

* Note that we could use different rulers for different lines but for convenience we use the same ruler for all lines.

Postulate 2 There is one and only one line trough two distinct given points (Existence and uniqueness)

Proposition 1 Two distinct lines have at most one point in common

Proof We are going to prove the proposition using the indirect method: ~~Ass~~

- Assume that the statement of the assertion is false
- Derive a contradiction to some known true statement or to one of the assumptions

Suppose that there exists two distinct lines with two points in common. This contradicts ~~there would be two~~ Postulate 2 that says that there can ~~be~~ only one.
The same argument works if the lines have 3, 4 ... points in common.

Definitions Given 3 points A, B, C

a) AB denotes the distance between A and B

b) If B lies on the line through A and C, then B is between A and C if the numbers corresponding to A, B, C occur in order

c) The part of the line through A and B containing A and B and all points in between is called the line segment AB (abusing the notation)

d) If A, B, C lie on the same line and $AB = BC$, B is said to bisect AC and B is the midpoint of AC

Note: the midpoint exists and it is unique as a consequence of Postulate 1

e) When two distinct lines have a point in common we say that they intersect and the point in common is the point of intersection.

f) Given a line ℓ and a point $P \in \ell$

p divides ℓ into 2 half-lines,

the half lines whose points are marked ~~with~~ with number greater than the mark at p and

the ...

If P and A are two points on a line ℓ,
we might use PA to denote the half-line starting at P containing A.

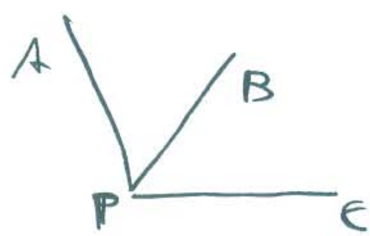
g) If 2 half-lines have the same end point then they are said to form 2 angles.

~~which~~ We usually denote the smaller angle by $\sphericalangle APB$ ~~or~~ angle APB or $\sphericalangle BPA$.

P is the vertex

PA, PB are the sides

If $\sphericalangle APB = \sphericalangle BPC$
then PB is said
to bisect $\sphericalangle APC$ and
PB is the bisector



Note we know what angles are and we know what it means for them to be equal. The next postulate measures them.

Postulate 3 (Principle of angle measure)

For each point P, there exists a bijection between the rays at P and the real numbers modulo 360° and this bijection respects angles.

In other words, if ω_{PR} and ω_{PS} are the numbers corresponding to the rays PR and PS, then

$$\angle RPS = \omega_s - \omega_r \mod 2\pi$$

Modulo 360° means that we consider equal 2 numbers if their difference is a multiple of 360°.

Remark we are using a protractor to measure angles

Notation if $\omega_{PR} = 90°$ and $\omega_{PS} = 60°$ then

$$\angle RPS = 90° - 60° = 30°$$

Def When 2 half-lines with a common end-point make a straight ~~angle~~ then each angle is called a straight angle.

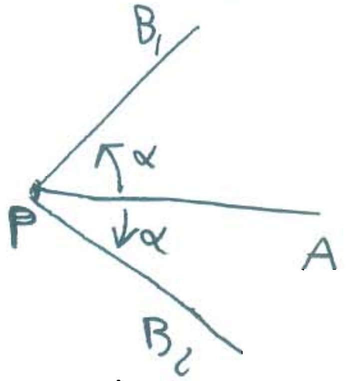
(Proposition\ Principle) A straight angle measures 180°

Def right angle 90°

acute angle $<90^\circ$

obtuse angle $>90^\circ$ but $<180^\circ$

Proposition Given a point P and a ray PA, for any $0<\alpha\neq 180^\circ$ there exists two distinct rays PB_1 and PB_2 s.t. $\angle APB_1 = \angle APB_2 = \alpha^\circ$



If $\alpha = 180^\circ$ then there exists a unique ray and it is a straight angle

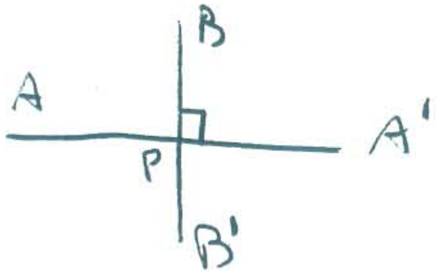
Proof The first part is a consequence of principle 3 ~~Using the bijection~~ Using the bijection, let PB_1 and PB_2 the rays associated to $\alpha = \omega_1$ and $\underset{\omega_2}{\underbrace{360 - \alpha}}$. Since $0\neq\alpha\neq 180$ $\omega_1 - \omega_2 = 2\alpha$ is not divisible by 360 which means PB_1 and PB_2 are distinct.

If $\alpha = 180°$ then the ray is clearly unique.
Let PB be such ray. Why A, P, B lie on the same line?

Let PC be ~~s.t. APC~~ the ray such that APC form a straight line.
From the proposition / principle $\angle APC = 180°$
Thus $\angle APC - \angle APB = 0$ which implies that PC and PB must be the same ray.

<u>Def</u> If 2 lines meet at a point P so that the angle of 2 of their half-lines measures 90° then the lines are said to be perpendicular.

<u>Problem</u>: What about the other angles?



<u>Prop</u> If one of the possible 4 angles is a right angle then all of them are.

Proof Suppose $\angle A'PB = 90^\circ$. Consider $\angle APB$. Since AA' is a straight line $\angle APA' = 180^\circ$

Therefore $\angle APB = \angle APA' - \angle BPA' = 180^\circ - 90^\circ = 90^\circ$

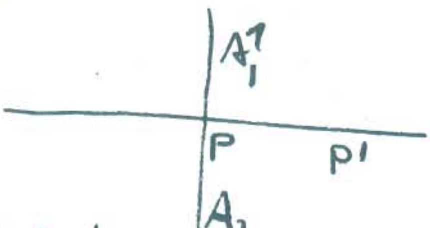
Same proof for the other 2

Consequence The notion of being perpendicular is independent of the choice of the two half-lines.

Proposition Given a point P on a line l there exists exactly one line perpendicular to l at P.

Proof Let PA_1 and PA_2 be the 2 unique and distinct rays such that

$\angle P'PA_1 = \angle P'PA_2 = 90^\circ$



Then $\angle A_1PA_2 = \angle A_1PP' + \angle A_2PP' = 180^\circ$

Which means that APA_2 form a line and because of the uniqueness of PA_1 and PA_2 such line is unique

Def. Let us consider a finite sequence of point $A_1, A_2, \dots A_n$

We say that the line segments $AA_2, A_2A_3, A_3A_4, \dots A_{n-1}A_n$ form the broken line $A_1A_2 \dots A_n$.

If $A_1 = A_n$ we call such broken line a polygon

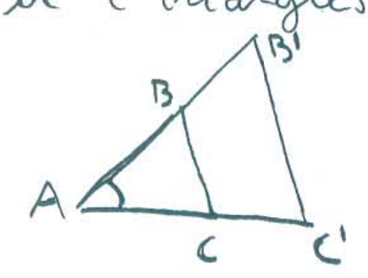
If $n=3$ we call such 3-sided polygon a triangle

Def Two polygons are similar if all corresponding angle are equals and all corresponding distances are proportionale.

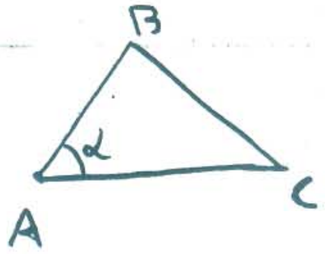
Examples Two triangles ABC and $A'B'C'$ are similar if $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} = k$

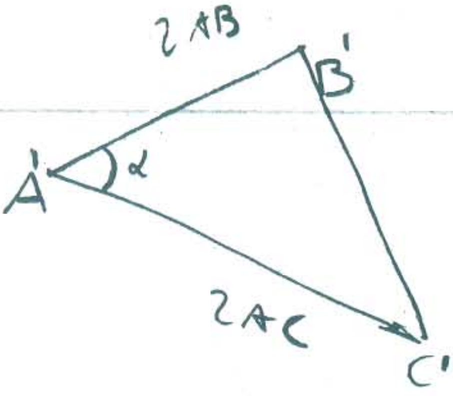
Principle 4 (Case 1 of similarity) (SAS)
Two triangle are similar if an angle of one equals an angle of the other and the sides including them are equal.
In other words, let ABC and $A'B'C'$ be 2 triangles then

if $\angle A = \angle A'$, $\frac{AB}{A'B'} = \frac{AC}{A'C'} = k$ $\Longrightarrow$ $\angle B = \angle B'$, $\angle C = \angle C'$, $\frac{BC}{B'C'} = k$



Consequence: Any given triangle can be reproduced anywhere either exactly or scaled.





Def Two similar triangle with proportionality factor equal to one are said to be congruent (or equal)

EQUAL IS A SUBJECTIVE CONCEPT.

Prove that the Cartesian plane is a model for Euclidean geometry.

$\mathbb{R}^2 = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

POINTS the set of pairs (x,y)

LINES the set of points which solve the eq $ax + by + c = 0$ where a or $b \neq 0$

DISTANCE $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

Summary:

4 PRINCIPLES

1. Line measure:
The points on any straight line can be numbered so that number differences measure distances

2. There is one and only one straight line through two given points

3. Angle measure
All half-lines having the same end-point can be numbered so that number differences measure angles

4. Case 1 of similarity
Two triangles are similar if an angle of one equals an angle of the other and the sides including the angle are proportionals.