1. Which ones of the following triples cannot represent the data of a triangle?

(1) $AB = 2, BC = 3, AC = 5$

(2) $AB = 11, BC = 13, AC = 15$

(3) $AB = 25, BC = 17, AC = 9$

(4) $AB = 20, BC = 35, AC = 17$

(5) $\angle A = 10°, \angle B = 10°, \angle C = 90°$

(6) $\angle A = 50°, \angle B = 30°, \angle C = 100°$

Solution: (1) and (5) because the sum of the lengths of 2 sides is always greater than the length of the third side and because the sum of the angles of a triangle is 180°.

2. Prove that the sum of the angles of a triangle is 180°.

Solution: in the notes or in the book.

3. Let ABC and $A'B'C'$ be triangles such that $\angle B = \angle B'$, $AB = 3A'B'$ and $BC = 3B'C'$. If $AC = 2$, what is the value of $A'C'$?

Solution: By the (SAS) criteria ABC and $A'B'C'$ are similar triangle and thus $AC/A'C' = BC/B'C' = 1/3$.

4. Explain why none of the following functions is an isometry:

(1) $4z$

(2) $\frac{z}{5}$

(3) $e^{|z|}$

Solution:

- $f(z) = 4z$: $f(0) = 0$ but $f(1) = 4$ thus $|f(1) - f(0)| = 4$ that is different from 1.
- $f(z) = \frac{z}{5}$: $f(0) = 0$ but $f(1) = \frac{1}{5}$ thus $|f(1) - f(0)| = \frac{1}{5}$ that is different from 1.
- $f(z) = e^{|z|}$: $f(1) = f(-1)$. The function is not even a bijection.

5. Let $ABCD$ be a trapezoid such that $AB \parallel DC$, $BD = 6$, $AB = 12$ and $\angle ADB = 90°$. What is $\angle ADC$?

Solution: ADB is half of an equilateral triangle.

$\angle ABD = 60°$ and $\angle BAD = 30°$.

By the properties of parallel lines $\angle BDC = \angle ABD = 60°$, therefore $\angle ADC = 150°$.

6. Prove that given a circle Γ and a point P outside of Γ there exist exactly two lines through P tangent to Γ.

Solution: HW 7 Ex 4.

7. Classify the following isometries of the Euclidean plane. In other words, which of the following isometries is a rotation, translation, line reflection or none of these?

(A) $-z+2$

(B) $iz+17$

(C) $z+5$

(D) $\bar{z}+2i$

(E) $i\bar{z}+2i$

Solution: Recall that an isometry is a rotation if and only if it has a unique fixed point. It is a reflection across a line if and only if it has a line of fixed points. Let $f(z)$ be an isometry, then its fixed points are solutions of the equation $z=f(z)$

(A) $f(z)=-z+2$: $z=-z+2$ gives that 1 is the unique fixed point. Thus it is a rotation.

(B) $f(z)=iz+17$: $z=iz+17$ gives $x+iy=ix-y+17$ that gives two equations. $x=-y+17$ and $y=x$. This implies that $f(z)$ has a unique fixed point. Thus it is a rotation.

(C) $f(z)=z+5$: $f(z)$ is a translation.

(D) $f(z)=\bar{z}+2i$: $z=\bar{z}+2i$ gives $x+iy=x-iy+2i$, that is $x=x$ and $y=1$. Thus, $f(z)$ has a line of fixed points and it is not the identity. $f(0)\neq 0$, thus it is a reflection across a line.

(E) $f(z)=i\bar{z}+2i$: $x+iy=ix+y+2i$ give $x=y$ and $y=x+2$. No fixed points. $f(0)=2i$ but $f(1)\neq 1+2i$ therefore it is not a translation.

8. Let Γ_1 and Γ_2 be two distinct circles centered at O_1 and O_2. Let l be a line tangent to Γ_1 and Γ_2 at P_1 and P_2 and suppose that O_1O_2 does not intersect l. Prove that $O_1O_2P_2P_1$ is a trapezoid.

Solution: Let $A=\Gamma_1\cap l$ and $B=\Gamma_2\cap l$. O_1A and O_2B are perpendicular to l and thus they are parallel.

9. Let ABC be a right triangle such that $\angle A = 90°$, $AB = 2$ and $AC = 3$. Suppose that ABC is inscribed in a circle of radius r. What is r?

Solution: Let O be the center of the theorem then $\angle BOC = 2\angle BAC = 180°$. Thus, B, O and C are collinear and BC must be the diameter of the circle. Since ABC is a right triangle, $2r = BC = \sqrt{4+9}$.

10. Recall that the general form of a Euclidean isometry is $az + b$ or $a\bar{z} + b$ where both a and b are complex numbers and $|a| = 1$. Recall that $z = x + iy$ where x and y are real numbers. Let $f(z)$ be the isometry which is the reflection across the line $y = 2x + 1$. Write down a formula for $f(z)$.

Solution: To determine an isometry, it suffices to find the image of 3 non collinear points.

Using complex notation $f(-\frac{1}{2}) = -\frac{1}{2}$, $f(i) = i$ and $f(0) = -\frac{4}{5} + i\frac{2}{5}$.

Now, let us impose this condition on the isometry. In either case, $f(0) = -\frac{4}{5} + i\frac{2}{5}$ implies that $b = -\frac{4}{5} + i\frac{2}{5}$.

Thus, the isometry is either $az - \frac{4}{5} + i\frac{2}{5}$ or $a\bar{z} - \frac{4}{5} + i\frac{2}{5}$.

$f(-\frac{1}{2}) = -\frac{1}{2}$ gives $-\frac{1}{2} = -a\frac{1}{2} - \frac{4}{5} + i\frac{2}{5}$ thus $a = -\frac{3}{5} + \frac{4}{5}i$.

Therefore, the isometry is either $(-\frac{3}{5} + \frac{4}{5}i)z - \frac{4}{5} + i\frac{2}{5}$ or $(-\frac{3}{5} + \frac{4}{5}i)\bar{z} - \frac{4}{5} + i\frac{2}{5}$.

It remains to use $f(i) = i$ to check which one it is. It is

$$(-\frac{3}{5} + \frac{4}{5}i)\bar{z} - \frac{4}{5} + i\frac{2}{5}.$$

11. Let ABC be an equilateral triangle, $AB = 5$, inscribed in a circle. What is the radius of the circle?

Solution: Let r be the radius, let O be the center of the circle and let AH be the height. AOB is an isosceles triangle ($OA = r = OB$).

Therefore $\angle ABO = \angle BAO = 30°$.

Thus OHB is a right triangle and $\angle HBO = 30°$.

This gives that OHB is half of an equilateral triangle and that $OH = \frac{OB}{2} = \frac{r}{2}$.

Finally, $AH = AO + OH = \frac{3}{2}r$.
Since ABC is an equilateral triangle, $AH = \frac{\sqrt{3}}{2}AB = \frac{5\sqrt{3}}{2}$.

Thus $\frac{5\sqrt{3}}{2} = \frac{3}{2}r$.

12. Let ABC be an isosceles triangle, $AB = AC$. Let M be a point on BC. Prove that M is the midpoint if and only if AM is perpendicular to BC.

Solution: Suppose M is the midpoint of BC, then by the (SSS) criteria AMB and AMC are similar triangles, in fact they are congruent.

Thus, $\angle AMB = \angle AMC$ and since their sum is a straight angle, they must be right angles. This proves $\implies$.

To prove the other implication, if AM is perpendicular to BC then $\angle AMB = \angle AMC$ and $\angle ABM = \angle ACM$ (it is a property of being isosceles).

Thus, by the AAA criteria AMB and AMC are similar triangle. In fact, since they have AM in common, they are congruent triangles.

Therefore, $AM = BM$.