Geometry I — Homework 9 — Due 16th Dec

1. Let l and m be two distinct lines and let T_l and T_m be the reflections across l and m. Prove that if l and m are parallel lines, then $T_l \circ T_m$ is a translation. Prove that otherwise $T_l \circ T_m$ is a rotation.

   Sol: Suppose first that l and m are parallel, let A and B be two points on m and let r be the distance between l and m. The reflection T_m leaves A and B fixed while $T_l(A) = T_l \circ T_m(A)$ is the point on the line perpendicular to l containing A such that $|A - T_l(A)| = 2r$ and the distance from $T_l(A)$ to l is r. The same thing is true of $T_l \circ T_m(B)$. Let C be the at distance $2r$ from l and r from m, then $T_m(C)$ is in l and $T_l \circ T_m(C) = T_m(C)$ is the point on the line perpendicular to l containing C such that $|C - T_l(C)| = 2r$. Since an isometry is uniquely determined by where it sends 3 non-collinear points, it is easy to conclude that $T_l \circ T_m$ is the translation that sends A into $T_l(A)$.

   If l and m intersect at a point O then, $T_l \circ T_m$ can only be a rotation (if O is the only fixed point), a reflection across a line or the identity. Like before, we are going to show that it is a rotation by studying where it sends 3 points. Let Γ be the circle of radius 1 centered at O and let L_1, L_2 and M_1, M_2 the points of intersection between Γ and l and m. Without loss of generality, suppose that $arcL_1M_1$ is shorter then $arcL_2M_1$. Then T_m does not move M_1 and T_l sends M_1 into $T_l(M_1) = T_l \circ T_m(M_1)$ such that the L_1 is the midpoint of $arcM_1T_l(M_1)$. In other words, $\angle M_1OT_l(M_1) = 2\angle M_1OL_1$. Let P be a point in the arc L_1M_2 such that M_2 is the midpoint of the arc PL_2. Then $T_m(P) = L_2$ and thus $T_lT_m(P) = T_l(L_2) = L_2$ Since, $\angle L_2OP = 2\angle M_2OL_2$ and $\angle M_2OL_2 = \angle M_1OL_1$ this shows that $T_l \circ T_m$ is a rotation around O of angle $2\angle M_1OL_1$.

2. Prove that the isometry $f(z) = \bar{z}$ is a reflection across a line.

   Sol: Since f is clearly not the identity, it suffices to show that it has more than one fixed point. Since real numbers are fixed points, we are done.

3. Prove that the isometry $f(z) = -iz + \sqrt{2}$ is a rotation. What is the angle of counter-clockwise rotation of f?

   Sol: It suffices to show that it has a unique fixed point. Thus, it follows easily after solving $z = -iz + \sqrt{2}$. The center of rotation is $\frac{\sqrt{2}}{1+i} = \frac{\sqrt{2}(1-i)}{2} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. Since $f(0) = \sqrt{2}$. the angle is $\frac{3}{2}\pi$.

4. Explain why neither of the following functions is an isometry: z^4, $10z$ and $|z - 2|$.

   Sol: $|z - 2|$ is not a bijection. The first two stretch the distance between say 0 and 2.