Geometry I — Homework 8 — Due 9th Dec

1. Prove that given two distinct points, A and B and an angle $\alpha \neq 0$, there exist exactly two rotations T_1 and T_2 of angle $|\alpha|$ such that $T_1(A) = B$ and $T_2(B) = A$. By convention, if α is positive then the rotation happens counterclockwise. If negative, clockwise.

   Sol: Let O be the center of such rotation. Then, the circle centered at O of radius OA must contain both A and B (this is because $B = T(A)$). Therefore AB is a chord of such circle which implies that O is on the perpendicular bisector of AB. Suppose $|\alpha| < 180°$. Let AD be the line that makes an angle $90° - |\frac{\alpha}{2}|$ and let D be its intersection with the perpendicular bisector. It is easy to show that this line must intersect the perpendicular bisector at O. Since there can only be two such lines, there can only be two such points on the perpendicular bisector. If $|\alpha| = 180°$ then O is the midpoint of AB.

2. Prove that an isometry sends a triangle into a congruent triangle.

   Sol: It follows by definition of isometry and (SSS).

3. Let Γ_1 and Γ_2 be two distinct circles centered at O_1 and O_2 and of radii r_1 and r_2, $r_1 \geq r_2$. Show that if $|O_1O_2| = r_1 + r_2$ or $|O_1O_2| = r_1 - r_2$, then the two circles are tangent, namely they intersect exactly at one point.

   Sol: If $|O_1O_2| = r_1 + r_2$ then the point P in O_1O_2 at distance r_1 from O_1 and r_2 from O_2 is a point in common. Suppose they have another point Q, different from P in common, then by the property of triangles, $O_1Q + O_2Q > O_1O_2 = r_1 + r_2$. Therefore, either $O_1Q > r_1$ or $O_2Q > r_2$. If $|O_1O_2| = r_1 - r_2$ then the point P on the line O_1O_2 at distance r_1 from O_1 and r_2 from O_2 is a point in common. Suppose they have another point Q, different from P in common, then by the property of triangles, $r_1 + QP = O_1Q + QP > O_1P = r_1$.

4. Given two distinct points A and B and an isometry T, prove that the line containing A and B is sent by T into the line containing $T(A)$ and $T(B)$.

   Sol: Let Q be a point on the line AB. If Q is between the points A and B then $|QA| + |QB| = |AB$. If $T(Q)$ is not on the line $T(A)T(B)$ then, $|QA| + |QB| = |T(Q)T(A)| + |T(Q)T(B)| > |T(A)T(B) = |AB$. If Q is not between the points A and B and, say, $|QA| > |QB|$. Then $|QA| = |AB| + |QB|$. If $T(Q)$ is not on the line $T(A)T(B)$ then, $|AB| + |QB| = |T(A)T(B)| + |T(Q)T(B)| > |T(Q)T(A) = |QA|$. Using the previous exercise, if a point Q is not on the line AB then the image of ABQ must be a triangle and therefore $T(A)$, $T(B)$ and $T(Q)$ can't be on the same line.

5. Prove that an isometry sends two parallel lines into two parallel lines.

   Sol: Let l and m be two parallel lines and let A and B two points in l and C and D two points in m. By the previous exercise, $T(l) = T(A)T(B)$ and $T(m) = T(C)T(D)$. Also, the line AC goes into the line $T(A)T(C)$. Use the fact that the triangles ABC and ACD are congruent to $T(ABC)$ and $T(ACD)$show that AC meets the lines l and m at the same angle, so does $T(A)T(C)$.