Geometry I — Homework 7 — Due 2nd Dec

1. Let ABC be a triangle inscribed in a circle and let P be a point in $arcBC$. Show that $\angle A = \angle PBC + \angle PCB$.

   Sol: Let O be the center of the circle. On the one hand, by Thales' Theorem, $\angle PBC = \frac{1}{2}\angle POC$ and $\angle PCB = \frac{1}{2}\angle POB$. On the other hand, again by Thales' Theorem, $\angle BAC = \frac{1}{2}BOC = \frac{1}{2}(\angle POC + \angle POB)$.

2. Let ABC be a triangle and let M in AB and N in AC such that CM bisects $\angle C$ and BN bisects $\angle B$. Let $D \in BN$ and $E \in CM$ such that $AD \perp BN$ and $AE \perp CM$. Prove that $DE \parallel CB$. (Hint: extend AD and AE to intersect the line containing BC in D' and E'. Show that AED and $AE'D'$ are similar.)

   Sol: Using (AAA) the triangle AEC and $E'EC$ are congruent. Giving that $AE' = 2AE$. Similarly, $AD' = 2AD$. Therefore, by (SAS), AED and $AE'D'$ are similar. In particular, $\angle AE'D' = \angle AED$ which implies $ED \parallel E'D'$.

3. Let l and m be two distinct lines intersecting at a point P. Prove that if a circle Γ is tangent to both lines then its center lies on one of the bisectors of the angles formed by such lines.

   Sol: Let O be the center of the circle and let $A = \Gamma \cap l$ and $B = \Gamma \cap m$. Then, AOP and BOP by (SSS) (they are right triangles with two equal sides).

4. Show that given a circle Γ and a point A outside Γ there exist exactly two lines containing A and tangent to Γ. (Hint: Let l be a tangent line, let O be the center of the circle, let $P \in \Gamma \cap l$ and let $B \in AO \cap \Gamma$. Let m be the tangent line to Γ at B. Studying $m \cap l$ prove first that there can only be two such lines (uniqueness). Then, find a way to construct them (existence).)

   Sol: Let $C = l \cap m$. ABC and AOB are similar by (AAA) (they are right triangle with a common angle). Therefore $\frac{BC}{AB} = \frac{OP}{AP}$. Since the length AP can be found using Pythagoras, the length BC is uniquely determined. Since there are only two points on m at this distance from B the statement follows. To construct the tangent lines then, draw the lines connecting A to these two points on m and show they must be tangent.