Geometry I — Homework 5 — Due 18th Nov

1. Let $ABCD$ be a quadrilateral. Prove that $ABCD$ is a parallelogram (opposite sides are parallel) if and only if opposite angles are equal.

   Sol: Recall that two lines are parallel if and only if a transversal meets them at equal angles (THM). Suppose first that $ABCD$ is a parallelogram, meaning $AB \parallel CD$ and $BC \parallel DA$, then since AC is transversal to AB and CD, $\angle CAB = \angle DCA$. Similarly, $\angle DAC = \angle ACB$. Therefore $\angle DAB = \angle DCB$. Similarly, $\angle ADC = \angle ABC$.

   Viceversa, suppose $\angle DAB = \angle DCB$ and $\angle ADC = \angle ABC$. Since the sum of the angle of a quadrilateral is 360° then $\angle DCB + \angle ABC = 180°$. Extend AB on the B side and let B' be a point on such extension. Then $\angle CBB' + \angle ABC = 180°$ which gives $\angle CBB' = \angle DCB$. Therefore $AB \parallel CD$...

2. Prove that the diagonals of a parallelogram $ABCD$ bisect each other.

   Sol: Let M be the intersection between AC and BD. Then, using THM and the AAA criteria for similar triangle one can show that AMB and DMC are similar.

3. Prove that if the diagonals of a quadrilateral $ABCD$ bisect each other then it is a parallelogram.

   Sol: Let M be the intersection between AC and BD. Then, SSS criteria for similar triangle one can show that AMB and DMC are similar. This implies that AC meets opposite sides of $ABCD$ at equal angles and therefore such sides are parallel.

4. A quadrilateral having two and only two sides parallel is a trapezoid. Let $ABCD$ be a trapezoid, $AD \parallel BC$ and let E and F be respectively the midpoint of AB and CD. Prove that EF is parallel to BC. (Hint: let H be the midpoint of AC. Show that EH and HF are parallel to BC. Conclude that E, H and F are collinear (on the same line) and that EF is parallel to BC.)

   Sol: CHF and CAD are similar by (SAS), therefore $\angle CFH = \angle CDA$. Thus, THM gives $HF \parallel AD \parallel BC$. Similarly, for EH. H can't be on two different parallels to BC therefore E, H and F and EF is parallel to BC.

5. Let ABC be a triangle and let BS be the bisector of $\angle B$ with S in AC. Prove that $AS/SC = AB/BC$. (Hint: extend BC in the direction of B and let R be the intersection between such extension and the parallel to BS through A.)

   Sol: Since AR and BS are parallel, $RB/AS = BC/SC$. Then, using THM, $\angle SAB = \angle BAS$ and $\angle ARB = \angle SBC$. Thus, $\angle ARB = \angle RAB$ giving that ABR is an isosceles triangle and $RB = AB$.