Geometry I — Homework 10 — Not Due

Recall:

- The general form of a Euclidean isometry is $az + b$ or $a\bar{z} + b$ where both a and b are complex numbers and $|a| = 1$.
- The equation of a hyperbolic line of the Poincaré disk which is not a diameter is $x^2 + y^2 - 2ax - 2by + 1 = 0$, where a and b are real number such that $a^2 + b^2 > 1$.

1. Find the isometries f that send the points 0, 1 and i, into

   (a) f(0)=1, f(1)=2, f(i)=i+1

   (b) f(0)=2i, f(1)=1+2i, f(i)=i

   (c) f(0)=0, f(1)=i, f(i)=-1

   Sol: The general form of an isometry is $az+b$ or $a\bar{z}+b$. First item: Using the first form for an Euclidean isometry, f(0)=1 gives $b = 1$, $f(1) = 2$ gives $a+b = 2$ and $f(i) = i+1$ gives $ai + b = i + 1$. Using the second form gives $b = 1$, $a + b = 2$ and $-ai + b = i + 1$. The first set of equations has a unique solution, $a = b = 1$ while the second set does not have a solution. Therefore, the first isometry must be of the first form and it is $f(z) = z + 1$. Arguing similarly, the second isometry is f(z)=$\bar{z}$+2i. The third is f(z)=iz

2. Find the equations of the hyperbolic lines of the Poincaré disk containing the points:

   (a) $(\frac{1}{4}, 0), (\frac{1}{2}, \frac{1}{2})$

   (b) $(\frac{1}{4}, \frac{3}{4}), (\frac{1}{5}, \frac{3}{5})$

   (c) $(\frac{1}{5}, \frac{1}{2}), (0, \frac{1}{3})$

   Sol: The second pair is on the line through the origin, $y = 3x$. The diameter contained in that line is the hyperbolic line. For the other 2, plug in the points in the equation $x^2 + y^2 - 2ax - 2by + 1 = 0$ and solve for a and b.

3. Prove that the isometry $f(z) = \bar{z} + 1$ is neither a rotation, nor a reflection across a line, nor a translation.

   Sol: It is not a rotation or a reflection because since $z = \bar{z} + 1$ does not have any solution, it has no fixed points. Since $f(0) = 1$, if it were a translation, then $f(i)$ must be $i + 1$ but it is not.

4. Prove that the isometry $f(z) = -\bar{z} + 2$ is a reflection across a line.

   Sol: It is not the identity and it has more than one fixed point, say 1 and i+1.