Non-proper complete minimal surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

Examples of complete minimal surfaces properly embedded in $\mathbb{H}^2 \times \mathbb{R}$ have been extensively studied and the literature contains a plethora of nontrivial ones. In this paper we construct a large class of examples of complete minimal surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$, not necessarily proper, which are invariant by a vertical translation or by a screw motion. In particular, we construct a large family of non-proper complete minimal disks embedded in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a vertical translation and a screw motion and whose importance is twofold. They have finite total curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by the isometry, thus highlighting a different behaviour from minimal surfaces embedded in $\mathbb{R}^3$ satisfying the same properties. They show that the Calabi-Yau conjectures for embedded minimal surfaces do not hold in $\mathbb{H}^2 \times \mathbb{R}$.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42

1 Introduction

Examples of complete minimal surfaces properly embedded in $\mathbb{H}^2 \times \mathbb{R}$ have been extensively studied and the literature contains a plethora of nontrivial ones. In this paper we focus on complete embedded examples, not necessarily proper, which are invariant by either a vertical translation, a screw motion (i.e. the composition of a rotation around a vertical straight line and a vertical translation) or a screw motion with axis at infinity (i.e. the composition of a horizontal parabolic translation and a vertical translation). The screw motions with axis at infinity are sometimes called parabolic screw motion (see [19]). Some examples with these properties, but all of them properly embedded, have been constructed in [10, 13, 14, 15, 16, 17, 19, 20].

The key examples contained in this paper are complete minimal disks embedded in $\mathbb{H}^2 \times \mathbb{R}$ that are non-proper and invariant by a vertical translation and a screw motion, we call them helicoidal-Scherk examples. The importance of such helicoidal-Scherk examples is twofold in understanding the behaviour of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

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In addition to being non-proper, a significant feature of these examples is that they have finite total curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by the vertical translation or the screw motion. In [21] Toubiana proved that a complete embedded minimal annulus with finite total curvature in the quotient of $\mathbb{R}^3$ by a translation must the quotient of a helicoid. In [11] Meeks and Rosenberg proved that Toubiana's result holds if the translation is replaced by a screw-motion. Moreover, in the same paper they also show that a complete embedded minimal surface with finite total curvature in the quotient of $\mathbb{R}^3$ by a translation or a screw-motion must be proper. Our examples highlight a much different behaviour in $\mathbb{H}^2 \times \mathbb{R}$. Recently, Collin, Hauswirth and Rosenberg have studied the conformal type and the geometry of the ends of properly embedded minimal surfaces with finite total curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a vertical translation [5]. Our main examples are related to but not included in their study.

The same examples are also of interest in relation to the Calabi-Yau conjectures for embedded minimal surfaces [1, 2, 22]. In [3], Colding and Minicozzi showed that a complete minimal surface embedded in $\mathbb{R}^3$ with finite topology is proper. See [12] for a generalization of their result. Our helicoidal-Scherk examples show that Colding and Minicozzi's result does not hold in $\mathbb{H}^2 \times \mathbb{R}$. Note that in [6] Coskunuzer has already constructed a complete embedded disk in $\mathbb{H}^3$ which is not proper, thus showing that Colding and Minicozzi's result does not generalize to $\mathbb{H}^3$. The techniques that we use to construct our examples are completely different from his.

The helicoidal-Scherk examples are constructed in the next section. In the other sections we further generalize the construction and also give examples of properly embedded minimal surfaces that are invariant by a screw motion with axis at infinity. These latter examples are included in the study in [5].

We would like to thank Laurent Hauswirth and Harold Rosenberg for very helpful conversations.

2 Helicoidal-Scherk examples

In order to construct our examples we consider the Poincaré disk model of $\mathbb{H}^2$; i.e.

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid |z| < 1\},$$

with the hyperbolic metric

$$g_{-1} = \frac{4}{(1-|z|^2)^2}|dz|^2.$$

We denote by $\partial_\infty \mathbb{H}$ the boundary at infinity of $\mathbb{H}^2$ and by $\mathbf{0}$ the origin of $\mathbb{H}^2$. We use t for the coordinate in $\mathbb{R}$. Finally, given any two points $p, q \in \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, we will denote by $\overline{pq}$ the geodesic arc joining them.

Let us consider $p_1 = 1$ and $p_2 = e^{i\frac{\pi}{2n}}$, for some $n \in \mathbb{N}$. Let Ω be the region bounded by the ideal geodesic triangle with vertices $\mathbf{0}, p_1$ and p_2 and edges $\overline{\mathbf{0}p_1}$, $\overline{\mathbf{0}p_2}$ and $\overline{p_1p_2}$. By Theorem 4.9 in [9], there exists a minimal graph over Ω with boundary values 0 over $\overline{\mathbf{0}p_1}$, h

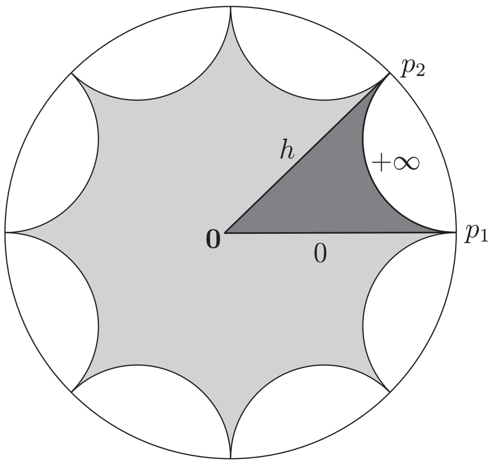

Figure 1: Fundamental piece of a helicoidal-Scherk example for n=2.

over $\overline{\mathbf{0}p_2}$ and $+\infty$ over $\overline{p_1p_2}$, for any constant $h > 0$ (see Figure 1). We call this graph the fundamental piece. Using Schwarz reflection principle, after considering successive symmetries with respect to the horizontal geodesics contained in the boundary of such a graph, namely $\overline{\mathbf{0}p_1}\times\{0\}, \overline{\mathbf{0}p_2}\times\{h\} \subset \mathbb{H}^2\times\mathbb{R}$, we obtain a simply-connected minimal surface $\widehat{M}_{n,h}$ with boundary the vertical line $\{\mathbf{0}\}\times\mathbb{R}$ and invariant by the vertical translation T by $(\mathbf{0}, 4nh)$ and by the screw motion S obtained by composing the rotation by angle $\frac{\pi}{n}$ around $\mathbf{0}$ and the vertical translation by $(\mathbf{0}, 2h)$. After reflecting across the line $\{\mathbf{0}\}\times\mathbb{R}$, we obtain a simply-connected complete minimal surface $M_{n,h}$ invariant by both the vertical translation T and the screw motion S. We call these surfaces *helicoidal-Scherk examples*.

We observe that if we consider $h = 0$ in this construction, we obtain a Scherk graph over a symmetric ideal polygonal domain, see [4, 16].

As a consequence of the Gauss-Bonnet Theorem applied on a fundamental piece (see [4, page 1896] for a similar argument), $M_{n,h}$ has finite total curvature in its quotient by both T and S.

We next show that $M_{n,h}$ is embedded. Let us denote by Ω_i, $i = 1, \cdots, 4n$, the domain obtained by rotating Ω around the origin by an angle $\frac{\pi}{2n}(i-1)$ so that $\Omega_1 = \Omega$, and let $\widetilde{\Omega}_i = \overline{\Omega}_i\times\mathbb{R}$, where $\overline{\Omega}_i$ is the closure of Ω_i. Note that the domain Ω_{2n+i} can also be obtained by reflecting Ω_i across the origin. We are going to prove that

$$M_{n,h} \cap \widetilde{\Omega}_{2n+1}$$

has no self-intersections. After this, repeating the same argument shows that $M_{n,h}$ is embedded. Let p_1^*, p_2^* be the reflection across the origin of p_1 and p_2. Recall that $\widehat{M}_{n,h} \cap \widetilde{\Omega}_1$ consists of a graph with boundary values 0 over $\overline{\mathbf{0}p_1}$, h over $\overline{\mathbf{0}p_2}$ and $+\infty$ over $\overline{p_1p_2}$, together with its vertical translates by the vector $k(\mathbf{0}, 4nh)$, $k \in \mathbb{Z}$. By construction, since we have reflected an even number of times, $\widehat{M}_{n,h} \cap \widetilde{\Omega}_{2n+1}$ consists of a union of graphs with boundary values

$$\begin{cases} 2nh + 4knh & \text{, over } \overline{\mathbf{0}p_1^*} \\ (2n+1)h + 4knh & \text{, over } \overline{\mathbf{0}p_2^*} \\ +\infty & \text{, over } \overline{p_1^* p_2^*} \end{cases}$$

with $k \in \mathbb{Z}$. The reflection of $\widehat{M}_{n,h} \cap \widetilde{\Omega}_1$ across $\{\mathbf{0}\} \times \mathbb{R}$ instead consists of a union of graphs with boundary values

$$\begin{cases} 4knh & \text{, over } \overline{\mathbf{0}p_1^*} \\ h + 4knh & \text{, over } \overline{\mathbf{0}p_2^*} \\ +\infty & \text{, over } \overline{p_1^* p_2^*} \end{cases}$$

with $k \in \mathbb{Z}$. In fact, $M_{n,h} \cap \widetilde{\Omega}_{2n+1}$ consists of the reflected fundamental piece together with its vertical translates by the vector $k(\mathbf{0}, 2nh)$, $k \in \mathbb{Z}$. In particular $M_{n,h} \cap \widetilde{\Omega}_{2n+1}$ is embedded. Repeating this argument proves that $M_{n,h}$ is embedded.

Observe that the previous argument also shows that we can consider the quotient of $M_{n,h}$ by $(\mathbf{0}, 2nh)$, obtaining a non-orientable complete non-proper embedded minimal surface.

Finally we remark that $M_{n,h} \cap \widetilde{\Omega}_1$ accumulates to $\overline{p_1 p_2} \times \mathbb{R}$, and therefore $M_{n,h}$ is a simply-connected (in particular, with finite topology) minimal surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ which is complete but not proper.

Let us now describe a generalization of these examples. Instead of considering a geodesic triangle, let Ω be the region bounded by an ideal geodesic polygon constructed in the following way. As before, let $p_1 = 1$ and $p_2 = e^{i\frac{\pi}{2n}}$ and let $\mathrm{arc}(p_1 p_2)$ denote the shortest arc in $\partial_\infty \mathbb{H}^2$ with end points p_1, p_2. In this construction, the geodesics $\overline{\mathbf{0}p_1}$ and $\overline{\mathbf{0}p_2}$ are the same but, instead of connecting the two with the geodesic $\overline{p_1 p_2}$, we consider $k \geq 1$ points $q_1, \cdots, q_k$, cyclically ordered in $\mathrm{arc}(p_1 p_2)$. We define Ω as the region bounded by the ideal geodesic polygon with vertices $\mathbf{0}, p_1, q_1, \cdots, q_k$ and p_2. Assuming that Ω satisfies the Jenkins-Serrin condition of Theorem 4.9 in [9], we can find a graph over Ω with boundary values 0 on $\overline{\mathbf{0}p_1}$, $h > 0$ on $\overline{\mathbf{0}p_2}$ and alternating $\pm\infty$ on the remaining geodesic arcs $\overline{p_1 q_1}$, $\overline{q_1 q_2}$, $\ldots$, $\overline{q_{k-1} q_k}$, $\overline{q_k p_2}$. After considering successive symmetries with respect to the horizontal and vertical geodesics contained in the boundary of such a graph, we obtain a simply-connected complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ invariant by the same vertical translation T and the same screw motion S previously defined. Again, this surface is non-proper and has finite total curvature when considered in the quotient by T or S. Moreover, it admits a non-orientable quotient by the vertical translation given by the vector $(\mathbf{0}, 2nh)$. We also refer to such examples as *helicoidal-Scherk examples*.

It is easy to show that the class of these more general examples is rather large. Here is an easy way to construct domains as previously described. If for $j = 1, \ldots, k$, we let $q_j = e^{i\frac{j\pi}{2n(k-1)}}$, then we recover the symmetric helicoidal-Scherk examples for a smaller choice of h, by the generalized maximum principle for such minimal graphs (see [4, Theorem 2] or [9, Theorem 4.13]). However, after slightly perturbing one such q_i, we would obtain a domain satisfying the Jenkins-Serrin condition of Theorem 4.9 in [9]. In particular, we observe that when $k = 1$ and q_1 is any point in $\mathrm{arc}(p_1p_2)$ then the Jenkins-Serrin condition is satisfied.

As mentioned in the introduction, the importance of these examples is twofold.

- In [21] Toubiana proved that a complete embedded minimal annulus with finite total curvature in the quotient of $\mathbb{R}^3$ by a translation must the quotient of a helicoid. In [11] Meeks and Rosenberg proved that Toubiana's result holds if the translation is replaced by a screw-motion. Moreover, in the same paper they also show that a complete embedded minimal surface with finite total curvature in the quotient of $\mathbb{R}^3$ by a translation or a screw-motion must be proper. Our examples highlight a much different behaviour in $\mathbb{H}^2 \times \mathbb{R}$.
- In [3], Colding and Minicozzi showed that a complete minimal surface embedded in $\mathbb{R}^3$ with finite topology is proper. Thus showing that the Calabi-Yau conjectures for complete embedded minimal surfaces in $\mathbb{R}^3$ hold, see [1, 2, 22]. Our helicoidal-Scherk examples show that Colding and Minicozzi's result does not hold in $\mathbb{H}^2 \times \mathbb{R}$. Note that in [6] Coskunuzer has already constructed a complete embedded disk in $\mathbb{H}^3$ which is not proper, thus showing that Colding and Minicozzi's result does not generalize to $\mathbb{H}^3$. The techniques that we have used to construct the helicoidal-Scherk examples are completely different from his. Note also that in [12], Meeks and Rosenberg generalized the result in [3] to complete minimal surfaces with positive injectivity radius and, among other things, showed that the closure of a complete minimal surface with positive injectivity radius embedded in a 3-manifold has the structure of a minimal lamination. The closure of a helicoidal-Scherk example is the minimal lamination given by the union of such helicoidal-Scherk example with the related totally geodesic vertical planes.

3 Helicoidal examples

Let us now consider $p_1 = 1$ and $p_2 = e^{i\frac{\pi}{m}}$, with $m \in \mathbb{N}$. Let Ω be the region bounded by $\overline{\mathbf{0}p_1}, \overline{\mathbf{0}p_2}$ and $\mathrm{arc}(p_1p_2)$, see Figure 2. By Theorem 4.9 in [9], there exists a minimal graph over Ω with boundary values 0 over $\overline{\mathbf{0}p_1}$, h over $\overline{\mathbf{0}p_2}$ and f over $\mathrm{arc}(p_1p_2)$, for any $h > 0$ and any continuous function f on $\mathrm{arc}(p_1p_2)$ (in fact, finitely many points of discontinuity for f are allowed). Again, after considering successive symmetries with respect to the horizontal geodesics contained in the boundary of such a graph, we get a minimal surface $\widehat{M}_{m,h,f}$ bounded by the vertical line $\{\mathbf{0}\} \times \mathbb{R}$ and invariant by the vertical translation T by $(\mathbf{0}, 2mh)$ and by the screw motion S obtained by composition of the rotation by angle $\frac{2\pi}{m}$ around $\mathbf{0}$ and the vertical translation by $(\mathbf{0}, 2h)$. Considering a final symmetry with respect to $\{\mathbf{0}\} \times \mathbb{R}$, we obtain a simply-connected complete

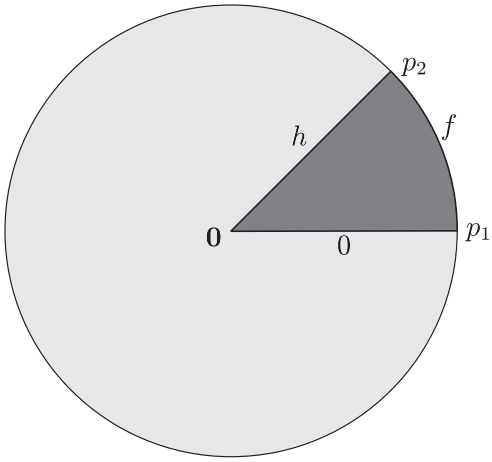

Figure 2: Fundamental piece of a helicoidal example with $m = 4$.

minimal surface $M_{m,h,f} \subset \mathbb{H}^2 \times \mathbb{R}$ which is invariant by the vertical translation T and by the screw motion S. We call these surfaces *helicoidal examples*.

Let us show that $M_{m,h,f}$ is embedded when m is even and, if f satisfies certain conditions, when m is odd. Using the same notation as in the previous section, we know that $\widehat{M}_{m,h,f} \cap \widetilde{\Omega}_{m+1}$ consists of the union of the minimal graph with boundary values

$$\begin{cases} mh & \text{, over } \overline{\mathbf{0}p_1^*} \\ (m+1)h & \text{, over } \overline{\mathbf{0}p_2^*} \\ f_m & \text{, over } \operatorname{arc}(p_1^* p_2^*) \end{cases}$$

where

$$f_m = \begin{cases} mh + f & \text{, if } m \text{ is even} \\ (1+m)h - f & \text{, if } m \text{ is odd} \end{cases}$$

together with its vertical translates by the vector $k(\mathbf{0}, 2mh)$, $k \in \mathbb{Z}$. The reflection of $\widehat{M}_{m,h,f} \cap \widetilde{\Omega}_1$ across $\{\mathbf{0}\} \times \mathbb{R}$ instead consists of the graph with boundary values

$$\begin{cases} 0 & \text{, over } \overline{\mathbf{0}p_1^*} \\ h & \text{, over } \overline{\mathbf{0}p_2^*} \\ f & \text{, over } \operatorname{arc}(p_1^* p_2^*) \end{cases}$$

together with its vertical translates by the vector $k(\mathbf{0}, 2mh)$, $k \in \mathbb{Z}$. Hence, using the general maximum principle for minimal graphs [9, Theorem 4.16], we get that $M_{m,h,f} \cap \widetilde{\Omega}_{m+1}$ is embedded when m is even or when m is odd and

$$(1-m)h \leq 2f \leq (1+m)h.$$

By symmetry, $M_{m,h,f}$ is embedded under the same conditions.

From the argument above we deduce that, when m is even, $M_{m,h,f} \cap \widetilde{\Omega}_{m+1}$ consists of the reflected fundamental piece together with its vertical translates by the vector $k(\mathbf{0}, mh)$, $k \in \mathbb{Z}$. Thus $M_{m,h,f}$ admits also a non-orientable quotient by $(\mathbf{0}, mh)$.

In the case m is even, if we consider the sequence of functions $\{f_k\}$, where $f_k = k$ over $\mathrm{arc}(p_1p_2)$, then we obtain the fundamental piece of the corresponding symmetric helicoidal-Scherk example as a limit of the fundamental piece of M_{m,h,f_k} and thus $M_{\frac{m}{2},h}$ as a limit of the sequence of surfaces $\{M_{m,h,f_k}\}_k$. We could take another choice of functions f_k with the same limit but in such a way that each M_{m,h,f_k} has a smooth boundary. In fact, any helicoidal-Scherk example can be recovered as a limit of some sequence $\{M_{m,h,f_k}\}_k$ of helicoidal examples, by choosing appropriate functions f_k.

Finally, observe that if we consider $f(e^{i\,t}) = \frac{hm}{\pi}\,t$, with $t \in (0, \frac{\pi}{m})$, we recover one of the helicoids given by Nelli and Rosenberg in [16], congruent to the Euclidean one. In fact, by varying $h > 0$ we re-obtain all of their examples. Hence the family of helicoidal examples contains the helicoids.

4 Helicoidal-Scherk examples with axis at infinity

We now take $p_0 = -1$, $p_1 = 1$ and $p_2 = e^{i\theta}$, for some $\theta \in (0, \pi)$. Let Ω be the region bounded by the ideal geodesic triangle with vertices p_0, p_1 and p_2, see Figure 3. By Theorem 4.9 in [9], there exists a minimal graph over Ω with boundary values 0 over $\overline{p_0p_1}$, h over $\overline{p_0p_2}$ and $+\infty$ over $\overline{p_1p_2}$, for any constant $h > 0$. After considering successive symmetries with respect to $\overline{p_0p_1} \times \{0\}$ and $\overline{p_0p_2} \times \{h\} \subset \mathbb{H}^2 \times \mathbb{R}$, we obtain a properly embedded minimal surface $M_{\theta,h}$ (in fact, it is a graph over an ideal polygonal domain with infinitely many boundary geodesic arcs) invariant by the screw motion P with axis at infinity, obtained by composition of the parabolic translation with fixed point p_0 which maps p_1 onto $e^{i2\theta}$ with the vertical translation by $(\mathbf{0}, 2h)$. We call these examples *helicoidal-Scherk examples with axis at infinity*, since they can be obtained as a limit of helicoidal-Scherk examples whose axes go to infinity. As a consequence of the Gauss-Bonnet Theorem, $M_{\theta,h}$ has finite total curvature in its quotient by P.

We observe that if we consider $h = 0$ in this construction, we obtain a pseudo-Scherk graph considered by Leguil and Rosenberg in [8].

Just like in section 2, these examples can be generalized by taking an ideal geodesic polygon Ω with vertices $p_0 = -1$, $p_1 = 1$, $p_2 = e^{i\theta}$ and $k \geq 1$ points $q_1, \cdots, q_k$ in $\mathrm{arc}(p_1p_2)$, such that Ω satisfies the Jenkins-Serrin condition of Theorem 4.9 in [9]. One such polygonal domain is called pseudo-Scherk polygon in [8]. We start with the graph over Ω with boundary values 0 on $\overline{p_0p_1}$, $h > 0$ on $\overline{p_0p_2}$ and alternating $\pm\infty$ on the remaining geodesics. After considering successive symmetries with respect to the horizontal geodesics contained in the boundary of such a graph, we obtain a properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ invariant by the screw motion P with axis at infinity described above (again, it is a graph over an ideal polygonal domain with

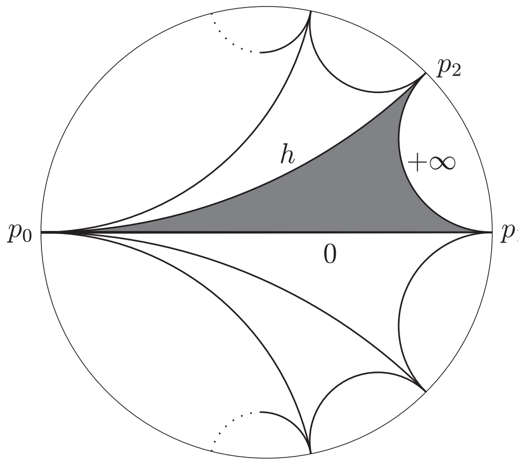

Figure 3: Fundamental piece of a helicoidal-Scherk example with axis at infinity.

infinitely many boundary geodesic arcs). In the quotient by P, such a surface has finite total curvature. We also refer to these generalized surfaces as *helicoidal-Scherk examples with axis at infinity.*

5 Helicoidal examples with axis at infinity

Let us now consider $p_0 = -1$, $p_1 = 1$ and $p_2 = e^{i\theta}$, with $\theta \in (0, \pi)$. Let Ω be the region bounded by $\overline{p_0p_1}, \overline{p_0p_2}$ and $\mathrm{arc}(p_1p_2)$, see Figure 4. By Theorem 4.9 in [9], there exists a minimal graph over Ω with boundary values 0 over $\overline{p_0p_1}$, h over $\overline{p_0p_2}$ and f over $\mathrm{arc}(p_1p_2)$, for any $h > 0$ and any continuous function f on $\mathrm{arc}(p_1p_2)$ (again, finitely many points of discontinuity for f are allowed). After considering successive symmetries with respect to the horizontal geodesics contained in the boundary of such a graph, we get a properly embedded minimal surface $M_{\theta,h,f}$ (in fact, it is an entire graph) invariant by the screw motion P with axis at infinity, obtained by composition of the parabolic translation with fixed point p_0 which maps p_1 onto $e^{i2\theta}$ with the vertical translation by $(\mathbf{0}, 2h)$. We call these surfaces *helicoidal examples with axis at infinity.*

We observe that, arguing as in section 3, any helicoidal-Scherk example with axis at infinity can be recovered as a limit of helicoidal examples with axis at infinity M_{θ,h,f_k}, by choosing appropriate functions f_k.

Finally, if we consider $f(e^{it}) = \frac{h}{\theta}\, t$, for any $t \in (0, \theta)$, we recover one of the examples invariant by the 1-parametric isometry group generated by P, founded by Onnis [17] and Sa Earp [19] independently.

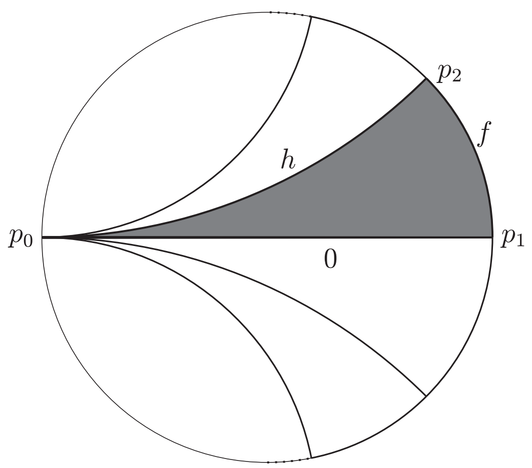

Figure 4: Fundamental piece of a helicoidal example with axis at infinity.

6 Non-periodic examples

In this last section, we point out how this method can be used to construct a lot of simply-connected examples which cannot be written as graphs. We now let $p_1 = 1$ and $p_2 = e^{i\theta}$, for some fixed $\theta \in (0, \pi)$, and define Ω as the domain bounded by $\overline{\mathbf{0}p_1}$, $\overline{\mathbf{0}p_2}$ and $\mathrm{arc}(p_1p_2)$. By Theorem 4.9 in [9], we know there exists a minimal graph over Ω with boundary values $+\infty$ on $\overline{\mathbf{0}p_1}$, 0 on $\overline{\mathbf{0}p_2}$ and f on $\mathrm{arc}(p_1p_2)$, for any continuous function f (again, finitely many discontinuity points are allowed). By rotating such a graph by an angle π about the horizontal geodesic $\overline{\mathbf{0}p_2} \times \{0\}$ contained in its boundary, we obtain a minimal graph whose boundary consists of the vertical line $\{\mathbf{0}\} \times \mathbb{R}$. After extending such a graph by symmetry about its boundary, we obtain a properly immersed simply-connected minimal surface. When $\theta \leq \pi/2$ or f is positive, the obtained surface is embedded. And its asymptotic boundary curve is smooth if $f = 0$ on p_2.

When $\theta \leq \pi/2$ and f diverges to $+\infty$ at any point (or to $\pm\infty$ alternately over a finite number of arcs contained in $\mathrm{arc}(p_1p_2)$, with some aditional restrictions to where the endpoints of such arcs are placed in order to satisfy the Jenkins-Serrin condition in the limit) we get the simply-connected minimal examples with finite total curvature constructed by Pyo and the first author in [18], called *twisted Scherk examples.*

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