Conformal Blocks: solutions to most of the exercises

2. Generalities on conformal transformations

2.1 Special conformal transformations

Problem:

Show that the general quadratic solution $\epsilon_\mu = \gamma_{\mu\nu\rho}x^\nu x^\rho$ to the equation $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \left(\frac{2}{d}\right)\delta_{\mu\nu}(\delta^{\sigma\tau}\partial_\sigma\epsilon_\tau)$, in flat d-dimensional Euclidean space is $\epsilon^\mu = 2x^\mu(b\cdot x) - b^\mu x^2$.

Solution

We start from the most general quadratic term $\epsilon_\mu = \gamma_{\mu\nu\rho}x^\nu x^\rho$, where $\gamma_{\mu\nu\rho} = \gamma_{\mu\rho\nu}$. (Any part of γ antisymmetric in the last two indiceswould not actually contribute anything to ϵ so we can take it to be zero).

We have to substitute this into the equation

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \left(\frac{2}{d}\right)\delta_{\mu\nu}(\delta^{\sigma\tau}\partial_\sigma\epsilon_\tau)\ . \qquad (*)$$

We first calculate (using $\gamma_{\tau\sigma\nu} = \gamma_{\tau\nu\sigma}$)

$$\begin{aligned} \partial_\sigma\epsilon_\tau &= \partial_\sigma(\gamma_{\tau\nu\rho}\,x^\nu x^\rho) = \gamma_{\tau\nu\rho}(\delta^\nu_\sigma x^\rho + x^\nu\delta^\rho_\sigma) = \gamma_{\tau\sigma\rho}x^\rho + \gamma_{\tau\nu\sigma}x^\nu = 2\gamma_{\tau\sigma\rho}x^\rho \\ \Rightarrow\ \delta^{\sigma\tau}\partial_\sigma\epsilon_\tau &= 2\gamma^\tau{}_{\tau\rho}\,x^\rho \end{aligned}$$

Since $\gamma^\tau{}_{\tau\rho}$ will occur quite a lot in the calculation, let's give it a name, $a_\rho = \gamma^\tau{}_{\tau\rho}$ and so we have

$$\delta^{\sigma\tau}\partial_\sigma\epsilon_\tau = 2a_\rho x^\rho\ .$$

Now the equation $(*)$ that we have to solve becomes

$$2\gamma_{\nu\mu\rho}x^\rho + 2\gamma_{\mu\nu\rho}x^\rho = \left(\tfrac{2}{d}\right)\delta_{\mu\nu}\,2a_\rho x^\rho$$

We now want to get rid of the x on the left hand side. We can do this by differentiating with respect to σ and using $\partial_\sigma x^\rho = \delta^\rho_\sigma$ to get

$$\Rightarrow\ 2\gamma_{\nu\mu\sigma} + 2\gamma_{\mu\nu\sigma} = \left(\tfrac{4}{d}\right)\delta_{\mu\nu}\,a_\sigma \qquad \Rightarrow\ \gamma_{\nu\mu\sigma} + \gamma_{\mu\nu\sigma} = \left(\tfrac{2}{d}\right)\delta_{\mu\nu}\,a_\sigma$$

This is just the sort of combination that occurs when calculating the Christoffel symbols in GR, and we use the same trick here: we take three version of this equation with different permuattions of the indices and use the symmetry on the last pair of indices:

$$\begin{aligned} \left\{\begin{array}{c} (\gamma_{\nu\mu\sigma} + \gamma_{\mu\nu\sigma}) \\ -(\gamma_{\mu\sigma\nu} + \gamma_{\sigma\mu\nu}) \\ +(\gamma_{\sigma\nu\mu} + \gamma_{\nu\sigma\mu}) \end{array}\right\} &= \left(\tfrac{2}{d}\right)\left\{\begin{array}{c} a_\sigma\delta_{\mu\nu} \\ -a_\nu\delta_{\mu\sigma} \\ +a_\mu\delta_{\nu\sigma} \end{array}\right\} \\ \Rightarrow\ 2\gamma_{\nu\mu\sigma} &= \left(\tfrac{2}{d}\right)(a_\sigma\delta_{\mu\nu} - a_\nu\delta_{\mu\sigma} + a_\mu\delta_{\nu\sigma}) \\ \Rightarrow\ 2\gamma_{\nu\mu\sigma} &= \left(\tfrac{2}{d}\right)(a_\sigma\delta_{\mu\nu} - a_\nu\delta_{\mu\sigma} + a_\mu\delta_{\nu\sigma}) \\ \epsilon_\mu = \gamma_{\mu\nu\rho}x^\nu x^\rho &= \left(\tfrac{1}{d}\right)(a_\rho x^\rho x_\sigma + a_\nu x^\nu x_\mu - a_\mu x^\rho x_\rho) \end{aligned}$$

Putting $a^\rho = d\,b^\rho$ we get the general solution

$$\epsilon^\mu = (2b^\sigma x_\sigma)x^\mu - b^\mu(x^2)\ .$$

Problem:

Consider the coordinate transformation of flat space

$$x'^{\mu} = \frac{x^{\mu} - (x^2)b^{\mu}}{1 - 2\,\mathbf{x}\cdot\mathbf{b} + x^2 b^2}\ , \tag{1}$$

where $\mathbf{b}$ is a constant vector. Show that $\frac{x'^{\mu}}{(x')^2} = \frac{x^{\mu}}{x^2} - b^{\mu}$.

Solution

We follow the hint and first find

$$(x')^2 = \frac{1}{(1-2\boldsymbol{x}\cdot\boldsymbol{b}+x^2b^2)^2}\left(x^2 - 2(\boldsymbol{x}\cdot\boldsymbol{b})x^2 + (x^2)^2b^2\right) = \frac{x^2}{(1-2\boldsymbol{x}\cdot\boldsymbol{b}+x^2b^2)}$$

so that

$$\frac{x'^{\mu}}{(x')^2} = \frac{(1-2\boldsymbol{x}\cdot\boldsymbol{b}+x^2b^2)}{x^2}\left[\frac{x^{\mu}-x^2b^{\mu}}{1-2\boldsymbol{x}\cdot\boldsymbol{b}+x^2b^2}\right] = \frac{x^{\mu}}{(x^2)} - b^{\mu}$$

Problem:

Consider now the vector $y^{\mu}(t)$ with real parameter t defined by $y(t)^{\mu} = \frac{x^{\mu}-(x^2)te^{\mu}}{1-2t\,\mathbf{x}\cdot\mathbf{e}+x^2t^2}$, where $\mathbf{e} = \hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$. We denote $|\mathbf{b}| = b$ so that $\mathbf{b} = b\,\mathbf{e}$. Show that

$$(a)\ \ \mathbf{e}\cdot\mathbf{y} = \frac{\mathbf{e}\cdot\mathbf{x} - tx^2}{1-2t\,\mathbf{x}\cdot\mathbf{e}+x^2t^2}\ ,(b)\ \ y^2 = \frac{x^2}{1-2t\,\mathbf{x}\cdot\mathbf{e}+x^2t^2}\ ,(c)\ \ \frac{\mathrm{d}y^{\mu}}{\mathrm{d}t} = 2(\mathbf{e}\cdot\mathbf{y})y^{\mu} - y^2\,e^{\mu}\ .$$

Solution:

(a)

$$\boldsymbol{e}\cdot\boldsymbol{y} = \boldsymbol{e}\cdot\left(\frac{\boldsymbol{x}-(x^2)t\boldsymbol{e}}{1-2t\,\boldsymbol{x}\cdot\boldsymbol{e}+x^2t^2}\right) = \frac{\boldsymbol{e}\cdot\boldsymbol{x}-(x^2)t\boldsymbol{e}\cdot\boldsymbol{e}}{1-2t\,\boldsymbol{x}\cdot\boldsymbol{e}+x^2t^2} = \frac{\boldsymbol{e}\cdot\boldsymbol{x}-(x^2)t}{1-2t\,\boldsymbol{x}\cdot\boldsymbol{e}+x^2t^2}$$

since $\boldsymbol{e}$ is a unit vectot, $\boldsymbol{e}\cdot\boldsymbol{e} = 1$.

(b) We have already done this in part (a) where we calculated $(x')^2$. If, for the moment, we substitute $\boldsymbol{b} = t\boldsymbol{e}$ then we get $x'^{\mu} = y^{\mu}$ and so, (using$\boldsymbol{e}\cdot\boldsymbol{e} = 1$)

$$y^2 = \frac{x^2}{(1-2\boldsymbol{x}\cdot(t\boldsymbol{e})+x^2(t\boldsymbol{e})^2)} = \frac{x^2}{(1-2t\boldsymbol{x}\cdot\boldsymbol{e}+x^2t^2)}$$

(c)

$$\frac{\mathrm{d}y^{\mu}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\boldsymbol{x}-(x^2)t\boldsymbol{e}}{1-2t\,\boldsymbol{x}\cdot\boldsymbol{e}+x^2t^2}\right] = \left[\frac{-(x^2)\boldsymbol{e}}{1-2t\,\boldsymbol{x}\cdot\boldsymbol{e}+x^2t^2}\right] - \left[\frac{(-2\,\boldsymbol{x}\cdot\boldsymbol{e}+2x^2t)(\boldsymbol{x}-(x^2)t\boldsymbol{e})}{(1-2t\,\boldsymbol{x}\cdot\boldsymbol{e}+x^2t^2)^2}\right]$$

$$= y^{\mu}\left[\frac{2(\boldsymbol{e}\cdot\boldsymbol{x}-x^2t)}{1-2t\boldsymbol{e}\cdot\boldsymbol{x}+x^2t^2}\right] - y^2y^{\mu} = 2(\boldsymbol{e}\cdot\boldsymbol{y})y^{\mu} - y^2\,e^{\mu}\ .$$

as required.

2.2 Classical scale invariant Lagrangians

Consider a scalar field which transforms under an infinitesimal coordinate transformation $x^\mu \to x'^\mu$ as $\phi(x) = |\partial x'^\mu/\partial x^\nu|^{\Delta/d}\,\phi(x')$, where $|\partial x'^\mu/\partial x^\nu|$ is the Jacobian of the transformation.

(a) Show that under a scale transformation $x'^\mu = \lambda x^\mu$, the field ϕ has scale dimension Δ.

(b) Show that if $\delta x^\mu = \alpha^\mu$, the variation of ϕ is $\delta\phi = \frac{\Delta}{d}(\partial_\mu\alpha^\mu)\phi \;+\; \alpha^\sigma\partial_\sigma\phi$.
(c) Show for an infinitesimal scale transformation $\delta x^\mu = \epsilon x^\mu$, $\delta_\epsilon\phi = \epsilon(\Delta\phi + x^\nu\partial_\nu\phi)$.
(d) Show that the variation of the Lagrangian density $\mathcal{L} = \frac{1}{2}\,\partial_\sigma\phi\partial^\sigma\phi - V(\phi)$, is a total derivative under an infinitesimal scale transformation provided $\Delta = (d/2) - 1$, and $V = c\,\phi^{D/\Delta}$ for some constant c. [What are these potentials?]

Solution:

(a) If $x'^\mu = \lambda x^\mu$ then $\partial x'^\mu/\partial x^\nu = \lambda\delta^\mu_\nu$, or $(\partial x'^\mu/\partial x^\nu) = \lambda\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix.

This means that

$$\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) \;=\; \det(\lambda\boldsymbol{I}) = \lambda^d \qquad \Rightarrow \quad \phi(x) = (\lambda^d)^{\Delta/d}\phi(x') = \lambda^\Delta\,\phi(x') \;.$$

We see that the field ϕ has scale dimension Δ.

(b) If $\delta x^\mu = \alpha^\mu$ then

$$x'^\mu \;=\; x^\mu + \delta x^\mu = x^\mu + \alpha^\mu \qquad \Rightarrow \frac{\partial}{\partial x^\nu}\,(x^\mu + \alpha^\mu) = \delta^\mu_\nu + \partial_\nu\alpha^\mu$$

We can write the Jacobian matrix of partial derivatives as

$$\frac{\partial}{\partial x^\nu}\,(x^\mu + \alpha^\mu) \;=\; \boldsymbol{I} + \boldsymbol{A}$$

where $\boldsymbol{I}$ is the identity matrix and the elements of the matrix $\boldsymbol{A}$ are $\partial\alpha^\mu/\partial x^\nu$. Since α^μ is an infinitesimal transformation, the entries in A are small and so

$$\det(\boldsymbol{I} + \boldsymbol{A}) \;=\; 1 + \mathrm{Tr}(\boldsymbol{A}) + O(A^2) = 1 + \partial_\mu\alpha^\mu$$

and thus

$$\left|\frac{\partial x'^\mu}{\partial x^\nu}\right|^{\Delta/d}\phi(x + \delta x) \;=\; \phi(x^\mu + \alpha^\mu)(1 + \partial_\nu\alpha^\nu + O(\alpha^2))^{\Delta/d}$$

$$= (\phi(x) + \alpha^\sigma\partial_\sigma\phi + O(\alpha^2))(1 + \frac{\Delta}{d}\partial_\nu\alpha^\nu + O(\alpha^2)) \;=\; \phi(x) + (\alpha^\sigma\partial_\sigma\phi + \frac{\Delta}{d}\partial_\mu\alpha^\mu\phi) + O(\alpha^2)$$

$$\Rightarrow \quad \boxed{\delta\phi = \frac{\Delta}{d}(\partial\cdot\alpha)\phi + \alpha\cdot\partial\phi}$$

(c) If we have an infinitesimal scale transformation $\delta x^\mu = \epsilon x^m u$, then

$$\partial\cdot\alpha = \partial_\mu(\epsilon x^\mu) = \epsilon\,\delta^\mu_\mu = \epsilon\,d\;, \quad \text{and} \quad \alpha\cdot\partial = \epsilon x^\mu\partial_\mu\;,$$

and so

$$\delta\phi = \epsilon(\Delta\phi + x^\nu\partial_\nu\phi)\;. \qquad (\dagger)$$

(d) We have

$$\delta\mathcal{L} = \partial_\sigma(\delta\phi)\partial^\sigma\phi - \delta\phi\frac{\partial V}{\partial\phi}$$

where we have written $\partial V/\partial\phi = V'$

$$\begin{aligned}
&= \Delta\partial_\sigma\phi\partial^\sigma\phi + \delta_\sigma^\tau\partial_\tau\phi\partial^\sigma\phi + x^\tau\partial_\tau\partial_\sigma\phi\partial^\sigma\phi - \Delta\phi V' - x^\tau\partial_\tau V \\
&= (\Delta+1)\partial_\sigma\phi\partial^\sigma\phi + \frac{1}{2}x^\tau\partial_\tau(\partial_\sigma\phi\partial^\sigma\phi) - \partial_\tau(x^\tau V) + (\partial_\tau x^\tau)V - \Delta\phi V' \\
&= \partial_\tau(\frac{1}{2}x^\tau\partial_\sigma\phi\partial^\sigma\phi - x^\tau V) + \left[\Delta + 1 - \frac{d}{2}\right](\partial\phi)^2 + (dV - \Delta\phi V') \\
&= \partial_\tau(x^\tau\mathcal{L}) + \left[\Delta - \frac{d-2}{2}\right](\partial\phi)^2 + (dV - \Delta\phi V')
\end{aligned}$$

This is a total derivative under an infinitesimal scale transformation provided $\Delta = (d/2) - 1$, and $dV = \Delta\phi V'$. This last equation is solved by $V = c\,\phi^{d/\Delta}$ for any constant c, as required.

$d/\Delta = \frac{2d}{d-2}$. This is an integer for $d = 3, 4, 6$ in which case the classically scale-invariant potentials are

d	3	4	6
V	ϕ^6	ϕ^4	ϕ^3

We note that these are the renormalisable potentials.

2.3 Scale invariance is not conformal invariance

Consider the following Lagrangian in four dimensions

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_A \ , \ \ \text{where} \ \ \mathcal{L}_\phi = \partial_\sigma \bar{\phi}\, \partial^\sigma \phi \ , \ \ \mathcal{L}_A = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \ ,$$

where ϕ is a complex scalar field, $\bar{\phi}$ is its conjugate and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of a gauge field A_μ. Under conformal transformations, the fields vary as

$$\delta\phi = \frac{1}{4}(\partial\cdot\epsilon)\phi + \epsilon^\sigma \partial_\sigma \phi \ , \ \ \delta\bar{\phi} = \frac{1}{4}(\partial\cdot\epsilon)\bar{\phi} + \epsilon^\sigma \partial_\sigma \bar{\phi} \ , \ \ \delta A_\mu = \epsilon^\sigma \partial_\sigma A_\mu + A_\sigma \partial_\mu \epsilon^\sigma \ .$$

(a) Show that

$$\delta\mathcal{L}_\phi = \partial_\mu \left[\epsilon^\mu (\partial\phi\cdot\partial\bar{\phi}) + \frac{1}{4}\phi\bar{\phi}\partial^\mu(\partial\cdot\epsilon) \right] - \frac{1}{4}\phi\bar{\phi}\partial_\sigma \Box \epsilon^\sigma + \left[\partial_\sigma\epsilon_\tau + \partial_\tau \epsilon_\sigma - \frac{1}{2}(\partial\cdot\epsilon)\eta_{\sigma\tau} \right] \partial^\sigma\phi\, \partial^\tau\bar{\phi} \ ,$$

hence $\mathcal{L}_\phi$ is invariant (up to a total derivative) for conformal transformations (explain why).

(b) Show that

$$\delta F_{\mu\nu} = (\partial_\mu \epsilon^\sigma) F_{\sigma\nu} + (\partial_\nu \epsilon^\sigma) F_{\mu\sigma} + \epsilon^\sigma \partial_\sigma F_{\mu\nu} \ .$$

$$\delta\mathcal{L}_A = \partial_\sigma\Big(-\frac{1}{4}\epsilon^\sigma F_{\mu\nu}F^{\mu\nu}\Big) - \tfrac{1}{2}\left[\partial_\sigma\epsilon_\tau + \partial_\tau\epsilon_\sigma - \frac{1}{2}(\partial\cdot\epsilon)\eta_{\sigma\tau} \right] F^\sigma{}_\nu F^{\tau\nu} \ ,$$

so again $\mathcal{L}_A$ is invariant (up to a total derivative) for conformal transformations.

(c) Now consider the usual interaction term (up to a factor of $-ie$)

$$\mathcal{L}_1 = A^\mu J_\mu \quad \text{where} \quad J_\mu = (\bar{\phi}\,\partial_\mu\phi - \phi\,\partial_\mu\bar{\phi}) \ .$$

Show that

$$\delta J_\mu = \tfrac{1}{2}(\partial\cdot\epsilon)J_\mu + \epsilon^\sigma\partial_\sigma J_\mu + J_\tau \partial_\mu \epsilon^\tau \ .$$

$$\delta\mathcal{L}_1 = \partial_\sigma(\epsilon^\sigma A\cdot J) + \left[\partial_\tau\epsilon_\mu + \partial_\mu\epsilon_\tau - \tfrac{1}{2}(\partial\cdot\epsilon)\eta_{\mu\tau}\right] A^\tau J^\mu \ ,$$

and hence the interaction term is (classically) invariant (up to total derivatives) under both scale transformations and special conformal transformations.

(d). Consider now the interaction term

$$\mathcal{L}_1 = A^\mu K_\mu \quad \text{where} \quad K_\mu = (\bar{\phi}\,\partial_\mu\phi + \phi\,\partial_\mu\bar{\phi}) \ .$$

Show that

$$\delta K_\mu = \tfrac{1}{2}(\partial\cdot\epsilon)K_\mu + \epsilon^\sigma\partial_\sigma K_\mu + K_\tau\partial_\mu\epsilon^\tau + \tfrac{1}{2}\phi\bar{\phi}\partial_\mu(\partial\cdot\epsilon) \ .$$

$$\delta\mathcal{L}_2 = \partial_\sigma(\epsilon^\sigma A\cdot K) + \left[\partial_\tau\epsilon_\mu + \partial_\mu\epsilon_\tau - \tfrac{1}{2}(\partial\cdot\epsilon)\eta_{\mu\tau}\right] A^\tau K^\mu + \tfrac{1}{2}\phi\bar{\phi}A^\sigma(\partial_\sigma\partial_\tau\epsilon^\tau) \ .$$

Hence, this term is invariant (up to total derivatives) for conformal transformations for which $\partial_\sigma\partial_\tau\epsilon^\tau = 0$, ie for translations, rotations, scale transformations but not special conformal transformations (explain why).

This solution is included on the next page, labelled 13 from an MSc course problem set.

13.

$$\delta\phi = \tfrac{1}{4}(\partial\cdot\epsilon)\phi + \epsilon^\sigma\partial_\sigma\phi \;,\quad \delta\bar\phi = \tfrac{1}{4}(\partial\cdot\epsilon)\bar\phi + \epsilon^\sigma\partial_\sigma\bar\phi ,$$

$$\delta A_\mu = \varepsilon^\sigma\partial_\sigma A_\mu + A_\sigma\partial_\mu\epsilon^\sigma$$

(a) $\mathcal{L}_\phi = \partial_\sigma\bar\phi\,\partial^\sigma\phi$

$$\Rightarrow \delta\mathcal{L}_\phi = \partial_\sigma(\delta\bar\phi)\partial^\sigma\phi + \partial_\sigma\bar\phi\,\partial^\sigma(\delta\phi)$$

$$= \partial_\sigma\left(\tfrac{1}{4}(\partial\cdot\epsilon)\bar\phi + \epsilon^\tau\partial_\tau\bar\phi\right)\partial^\sigma\phi + \partial_\sigma\bar\phi\,\partial^\sigma\left(\tfrac{1}{4}(\partial\cdot\epsilon)\phi + \epsilon^\tau\partial_\tau\phi\right)$$

$$= \tfrac{1}{4}\left[\bar\phi(\partial^\sigma\phi)\partial_\sigma(\partial\cdot\epsilon) + \phi(\partial^\sigma\bar\phi)\partial_\sigma(\partial\cdot\epsilon)\right] + \tfrac{1}{4}(\partial\cdot\epsilon)(\partial_\sigma\bar\phi\partial^\sigma\phi + \partial_\sigma\bar\phi\partial^\sigma\phi)$$
$$+ \partial_\sigma\epsilon^\tau(\partial_\tau\bar\phi\,\partial^\sigma\phi) + \epsilon^\tau\left[\partial_\tau(\partial_\sigma\bar\phi)\partial^\sigma\phi + \partial_\sigma\bar\phi\,\partial_\tau(\partial^\sigma\phi)\right] + \partial_\sigma\epsilon^\tau(\partial^\sigma\bar\phi\,\partial_\tau\phi)$$

$$= \tfrac{1}{4}\partial^\sigma(\bar\phi\phi)\partial_\sigma(\partial\cdot\epsilon) + \tfrac{1}{2}(\partial\cdot\epsilon)(\partial_\sigma\bar\phi\partial^\sigma\phi) + (\partial_\sigma\epsilon_\tau + \partial_\tau\epsilon_\sigma)(\partial^\tau\bar\phi\,\partial^\sigma\phi) + \epsilon^\tau\partial_\tau(\partial_\sigma\bar\phi\partial^\sigma\phi)$$

$$= \tfrac{1}{4}\partial^\sigma(\bar\phi\phi\,\partial_\sigma(\partial\cdot\epsilon)) - \tfrac{1}{4}\bar\phi\phi\,\Box(\partial\cdot\epsilon) + \partial_\tau(\epsilon^\tau(\partial_\sigma\bar\phi\partial^\sigma\phi)) - (\partial\cdot\epsilon)(\partial_\sigma\bar\phi\,\partial^\sigma\phi) + \tfrac{1}{2}(\partial\cdot\epsilon)(\partial_\sigma\bar\phi\partial^\sigma\phi)$$
$$+ (\partial_\sigma\epsilon_\tau + \partial_\tau\epsilon_\sigma)(\partial^\tau\bar\phi\,\partial^\sigma\phi)$$

$$= \partial_\sigma\left(\tfrac{1}{4}\bar\phi\phi(\partial^\sigma(\partial\cdot\epsilon)) + \varepsilon^\sigma\partial\bar\phi\cdot\partial\phi\right) - \tfrac{1}{4}\bar\phi\phi\,\Box(\partial\cdot\epsilon)$$
$$+ (\partial_\sigma\epsilon_\tau + \partial_\tau\epsilon_\sigma - \tfrac{1}{2}(\partial\cdot\epsilon)\eta_{\tau\sigma})\,\partial^\tau\bar\phi\partial^\sigma\phi$$

This is a total divergence for conformal transformations in 4 dimensions as

(1) $\partial_\sigma\epsilon_\tau + \partial_\tau\epsilon_\sigma = \frac{2}{d}\eta_{\tau\sigma}(\partial\cdot\epsilon) = \frac{1}{2}\eta_{\tau\sigma}(\partial\cdot\epsilon)$

(2) $\partial_\sigma\partial_\tau\partial_\rho\epsilon_\nu = 0 \Rightarrow \Box(\partial\cdot\epsilon) = 0$

(b) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\delta(F_{\mu\nu}) = \partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu)$$

$$= \partial_\mu(\epsilon^\sigma \partial_\sigma A_\nu + A_\sigma \partial_\nu \epsilon^\sigma) - \partial_\nu(\epsilon^\sigma \partial_\sigma A_\mu + A_\sigma \partial_\mu \epsilon^\sigma)$$

$$= (\partial_\mu \epsilon^\sigma)\partial_\sigma A_\nu + \epsilon^\sigma \partial_\sigma \partial_\mu A_\nu$$

$$+ (\partial_\mu A_\sigma)\partial_\nu \epsilon^\sigma + \cancel{A_\sigma \partial_\mu \partial_\nu \epsilon^\sigma}$$

$$- \partial_\nu \epsilon^\sigma \partial_\sigma A_\mu - \epsilon^\sigma \partial_\sigma \partial_\nu A_\mu$$

$$- (\partial_\nu A_\sigma)\partial_\mu \epsilon^\sigma - \cancel{A_\sigma \partial_\nu \partial_\mu \epsilon^\sigma}$$

$$= (\partial_\mu \epsilon^\sigma)(\partial_\sigma A_\nu - \partial_\nu A_\sigma)$$

$$+ (\partial_\nu \epsilon^\sigma)(\partial_\mu A_\sigma - \partial_\sigma A_\mu)$$

$$+ \epsilon^\sigma \partial_\sigma(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\Rightarrow \delta F_{\mu\nu} = \partial_\mu \epsilon^\sigma F_{\sigma\nu} + \partial_\nu \epsilon^\sigma F_{\mu\sigma} + \epsilon^\sigma \partial_\sigma F_{\mu\nu}$$

$$\Rightarrow \delta \mathcal{L}_A = \delta\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$$

$$= -\frac{1}{2} F^{\mu\nu} \delta(F_{\mu\nu})$$

$$= -\frac{1}{2} F^{\mu\nu}(\partial_\mu \epsilon^\sigma \cdot F_{\sigma\nu} + \partial_\nu \epsilon^\sigma \cdot F_{\mu\sigma} + \epsilon^\sigma \partial_\sigma F_{\mu\nu})$$

$$= -\frac{1}{4}\epsilon^\sigma \partial_\sigma (F^{\mu\nu} F_{\mu\nu}) - \frac{1}{2} F^\mu{}_\nu F^{\sigma\nu}(\partial_\mu \epsilon_\sigma + \partial_\sigma \epsilon_\mu)$$

$$= -\frac{1}{4}\partial_\sigma(\epsilon^\sigma F^{\mu\nu} F_{\mu\nu}) + \frac{1}{4}(\partial_\sigma \epsilon^\sigma) F^{\mu\nu} F_{\mu\nu}$$
$$- \frac{1}{2} F^\mu{}_\nu F^{\sigma\nu}(\partial_\mu \epsilon_\sigma + \partial_\sigma \epsilon_\mu)$$

$$= \underbrace{-\frac{1}{4}\partial_\sigma(\epsilon^\sigma F^{\mu\nu} F_{\mu\nu})}_{\partial_\sigma(\epsilon^\sigma \mathcal{L}_A)} - \frac{1}{2} F^\mu{}_\nu F^{\sigma\nu}\underbrace{\left(\partial_\mu \epsilon_\sigma + \partial_\sigma \epsilon_\mu - \frac{1}{2}\,{}_{\mu\sigma}\,\partial\cdot\epsilon\right)}$$

This vanishes if ϵ^μ is a conformal transformation in $d=4$.

(c) $\mathcal{L}_1 = A^\mu J_\mu \quad , \quad J_\mu = \bar\Phi \partial_\mu \Phi - \Phi \partial_\mu \bar\Phi$

$$\delta J_\mu = \delta\bar\Phi\, \partial_\mu\Phi + \bar\Phi \partial_\mu(\delta\Phi) - \delta\Phi(\partial_\mu\bar\Phi) - \Phi\partial_\mu(\delta\bar\Phi)$$

$$= \tfrac{1}{4}(\partial\cdot\epsilon)\,\bar\Phi\partial_\mu\Phi^{(1)} + \bar\Phi\partial_\mu\left(\tfrac14(\partial\cdot\epsilon)\Phi\right)^{(2} - \tfrac14(\partial\cdot\epsilon)\Phi\partial_\mu\bar\Phi^{(1)}$$
$$- \Phi\partial_\mu\left(\tfrac14(\partial\cdot\epsilon)\bar\Phi\right)^{(3}$$
$$+ \epsilon^\sigma\partial_\sigma\Phi\cdot\partial_\mu\bar\Phi + \bar\Phi\partial_\mu(\epsilon^\sigma\partial_\sigma\Phi) - \epsilon^\sigma\partial_\sigma\Phi\cdot\partial_\mu\bar\Phi - \Phi\partial_\mu(\epsilon^\sigma\partial_\sigma\bar\Phi)$$

$$= \tfrac14(\partial\cdot\epsilon)J_\mu^{(1)} + \tfrac14\partial_\mu(\partial\cdot\epsilon)\bar\Phi\Phi + \tfrac14(\partial\cdot\epsilon)\bar\Phi\partial_\mu\Phi\,]\,2$$
$$- \tfrac14\partial_\mu(\partial\cdot\epsilon)\bar\Phi\Phi - \tfrac14(\partial\cdot\epsilon)\Phi\partial_\mu\bar\Phi\,]\,3$$
$$+ \epsilon^\sigma\partial_\sigma(\Phi\partial_\mu\bar\Phi)^{L3} + \bar\Phi\partial_\mu\epsilon^\sigma\cdot\partial_\sigma\Phi^{L4}$$
$$- \epsilon^\sigma\partial_\sigma(\bar\Phi\partial_\mu\Phi)^{L3} - \Phi\partial_\mu\epsilon^\sigma\cdot\partial_\sigma\bar\Phi^{L4}$$

$$= \tfrac12(\partial\cdot\epsilon)J_\mu + \underbrace{\epsilon^\sigma\partial_\sigma J_\mu}_{3} + \underbrace{J_\sigma\partial_\mu\epsilon^\sigma}_{4}$$

$$\Rightarrow \delta\mathcal{L}_1 = \delta A^\mu\cdot J_\mu + A^\mu\cdot\delta J_\mu$$
$$= \epsilon^\sigma\partial_\sigma A^\mu\cdot J_\mu^{L1} + A_\sigma\partial^\mu\epsilon^\sigma\cdot J_\mu^{<2}$$
$$+ \tfrac12(\partial\cdot\epsilon)A^\mu J_\mu + A^\mu\epsilon^\sigma\partial_\sigma J_\mu^{L1} + A^\mu J_\sigma\partial_\mu\epsilon^{\sigma\,<3}$$
$$= \epsilon^\sigma\partial_\sigma(A^\mu J_\mu)^{L1} + (A_\sigma J_\mu)(\partial^\mu\epsilon^\sigma + \partial^\sigma\epsilon^\mu)$$
$$+ \tfrac12(\partial\cdot\epsilon)A^\mu J_\mu$$
$$= \partial_\sigma(\epsilon^\sigma A^\mu J_\mu) - A^\mu J_\mu(\partial\cdot\epsilon) + \tfrac12 A^\mu J_\mu(\partial\cdot\epsilon)$$
$$+ A_\sigma J_\mu(\partial^\mu\epsilon^\sigma + \partial^\sigma\epsilon^\mu)$$
$$= \underbrace{\partial_\sigma(\epsilon^\sigma\mathcal{L}_1)}_{\text{Total Derivative}} + A^\sigma J^\mu\underbrace{\left(\partial_\mu\epsilon_\sigma + \partial_\sigma\epsilon_\mu - \tfrac12\eta_{\mu\sigma}(\partial\cdot\epsilon)\right)}_{\text{Vanishes if } \epsilon^\mu \text{ is a conformal transformation.}}$$

(d) $K_\mu = \bar\Phi \partial_\mu \phi + \phi \partial_\mu \bar\Phi$

$$\delta K_\mu = \delta\bar\Phi \cdot \partial_\mu \phi + \bar\Phi \partial_\mu (\delta\phi) + \delta\phi \cdot \partial_\mu \bar\Phi + \phi \cdot \partial_\mu (\delta \bar\Phi)$$

$$= \frac{1}{4}(\partial\cdot\epsilon)\bar\Phi\partial_\mu\phi + \frac{1}{4}\bar\Phi\partial_\mu[(\partial\cdot\epsilon)\phi] + \frac{1}{4}(\partial\cdot\epsilon)\phi\partial_\mu\bar\Phi + \frac{1}{4}\phi\partial_\mu[(\partial\cdot\epsilon)\bar\Phi]$$

$$+ \epsilon^\sigma\partial_\sigma\bar\Phi\cdot\partial_\mu\phi + \bar\Phi\partial_\mu(\epsilon^\sigma\partial_\sigma\phi) + \epsilon^\sigma\partial_\sigma\phi\cdot\partial_\mu\bar\Phi + \phi\,\partial_\mu(\epsilon^\sigma\partial_\sigma\bar\Phi)$$

$$= \frac{1}{2}(\partial\cdot\epsilon)\bar\Phi\partial_\mu\phi + \frac{1}{4}\bar\Phi\phi\,\partial_\mu(\partial\cdot\epsilon) + \frac{1}{2}(\partial\cdot\epsilon)\phi\partial_\mu\bar\Phi + \frac{1}{4}\bar\Phi\phi\,\partial_\mu(\partial\cdot\epsilon)$$

$$+ \epsilon^\sigma\partial_\sigma(K_\mu) + \bar\Phi\partial_\sigma\phi\cdot\partial_\mu\epsilon^\sigma + \phi\partial_\sigma\bar\Phi\cdot\partial_\mu\epsilon^\sigma$$

$$= \frac{1}{2}(\partial\cdot\epsilon)K_\mu + \epsilon^\sigma\partial_\sigma K_\mu + K_\sigma\partial_\mu\epsilon^\sigma + \underbrace{\frac{1}{2}\bar\Phi\phi\,\partial_\mu(\partial\cdot\epsilon)}_{\text{extra term.}}$$

$$\mathcal{L}_2 = A^\mu K_\mu$$

$$\delta\mathcal{L}_2 = \delta A^\mu\cdot K_\mu + A^\mu\cdot\delta K_\mu$$

$$= (\epsilon^\sigma\partial_\sigma A^\mu \,{}^{<1} + A^\sigma\partial^\mu\epsilon^\sigma \,{}^{(2})K_\mu$$

$$+ A^\mu\left(\frac{1}{2}(\partial\cdot\epsilon)K_\mu + \epsilon^\sigma\partial_\sigma K_\mu \,{}^{<1} + K_\sigma\partial_\mu\epsilon^\sigma \,{}^{(3}\right) + \frac{1}{2}A^\mu\bar\Phi\phi(\partial_\mu\partial\cdot\epsilon)$$

$$= \epsilon^\sigma\partial_\sigma(A^\mu K_\mu) + A^\sigma K^\mu(\partial_\mu\epsilon_\sigma \,{}^{(2} + \partial_\sigma\epsilon_\mu \,{}^{(3})$$

$$+ \frac{1}{2}(\partial\cdot\epsilon)A^\mu K_\mu + \frac{1}{2}A^\mu\bar\Phi\phi\,\partial_\mu(\partial\cdot\epsilon)$$

$$= \underbrace{\partial_\sigma(\epsilon^\sigma A^\mu K_\mu)}_{\text{Total derivative}} + A^\sigma K^\mu\overbrace{\left(\partial_\mu\epsilon_\sigma + \partial_\sigma\epsilon_\mu - \frac{1}{2}\eta_{\sigma\mu}(\partial\cdot\epsilon)\right)}^{\text{vanishes for conf. transf.}}$$

$$+ \frac{1}{2}A^\mu\bar\Phi\phi\underbrace{\partial_\mu(\partial\cdot\epsilon)}_{\substack{\text{Vanishes for}\\ \text{Translations, Lorentz, \& Scale transf. but \underline{not}}\\ \text{Special conf. trans.}}}$$

ϵ^μ for Translations, Lorentz + ~~Special conf~~ scale is Linear
$\Rightarrow$ $\partial_\sigma\partial_\rho\epsilon^\mu = 0$. ϵ^μ for special conf is quadratic & this doesn't vanish.

3. Specialisation to $d = 2$

3.1 Conformal transformations in two dimensions

We have to consider the equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \delta_{\mu\nu} (\delta^{\sigma\tau} \partial_\sigma \epsilon_\tau) \ ,$$

for each choice of indices $\{\mu, \nu\}$ in Cartesian coordinates, that is for $\{\mu, \nu\} = \{x, x\}, \{x, y\}, \{y, y\}$.

$\{\mu, \nu\} = \{x, x\}$:

$$2\partial_x \epsilon_x = (\partial_x \epsilon_x + \partial_y \epsilon_y) \quad \Rightarrow \quad \partial_x \epsilon_x = \partial_y \epsilon_y$$

$\{\mu, \nu\} = \{x, y\}$:

$$\partial_x \epsilon_y + \partial_y \epsilon_x = 0$$

$\{\mu, \nu\} = \{y, y\}$:

$$2\partial_y \epsilon_y = (\partial_x \epsilon_x + \partial_y \epsilon_y) \quad \Rightarrow \quad \partial_y \epsilon_y = \partial_x \epsilon_x$$

If we now label the two components of ϵ as $\epsilon_x = f$ and $\epsilon_y = g$ we see that we have the following equations:

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \ , \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x} \ . \tag{$*$}$$

These are the Cauchy-Riemanns equations for the f and g to be the real and imaginary parts of a complex function of $z = x + iy$. If we put

$$F = f + ig \ , \quad \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y) \ , \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \ .$$

The equation that F is a differentiable function of z is

$$0 = \frac{\partial F}{\partial \bar{z}} = \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} \right) + i \left(\frac{\partial f}{\partial y} + \frac{\mathrm{d} g}{\partial x} \right) \ ,$$

which are exactly the equations $(*)$.

3.2 Special conformal transformations of the complex plane

We start from

$$x'^{\mu} = \frac{x^{\mu} - (x^2)b^{\mu}}{1 - 2\,\mathbf{x}\cdot\mathbf{b} + x^2 b^2}\ , \tag{1}$$

In complex coordinates $z = x+iy, \bar{z} = x - iy$ we have $\mathbf{x}\cdot\mathbf{x} = z\bar{z} \Rightarrow g_{z\bar{z}} = g_{\bar{z}z} = 1/2, g_{zz} = g_{\bar{z}\bar{z}} = 0$. Hence with $x^{\mu} = (z, \bar{z}), b^{\mu} = (b, \bar{b})$,

$$z' = \frac{z - (z\bar{z})b}{1 - (z\bar{b} + \bar{z}b) + z\bar{z}b\bar{b}} = \frac{z(1-\bar{z}b)}{(1-z\bar{b})(1-\bar{z}b)} = \frac{z}{1 - z\bar{b}}\ .$$

Similarly, or by complex conjugation, we get the transformation for $\bar{z}$,

$$z \mapsto \frac{z}{1-\bar{b}z}\ ,\ \ \bar{z} \mapsto \frac{\bar{z}}{1 - b\bar{z}}\ .$$

If $b = \alpha + i\beta$ then

$$\delta z = \delta x + i\delta y = (\alpha+i\beta)(x+iy)^2 = (\alpha+i\beta)(x^2-y^2+2ixy) = (\alpha(x^2-y^2)-2\beta xy) + i(\beta(x^2-y^2)+2\alpha xy)\ ,$$

generated by $Q = (\alpha+i\beta)L_1 + (\alpha - i\beta\bar{L}_1) = \alpha(L_1+\bar{L}_1) + \beta i(L_1 - \bar{L}_1)$.

3.3 Möbius maps

A Möbius map is of the form

$$z \mapsto \frac{az+b}{cz+d}$$

The simplest way to show there is a unique map that sends $u \to u', v \to v', w \to w'$ is to show there is a unique map that sends $u \to \infty, v \to 1, w \to 0$ and then there is a unique composition that sends $u \to \infty \to u'$, etc.

Obviously to send $u \to \infty$ and $w \to 0$ we must have

$$z \mapsto A\,\frac{z-w}{z-u}$$

for some constant A and we choose A to send $v \to 1$,

$$z \mapsto \frac{v-u}{v-w}\,\frac{z-w}{z-u}$$

A Möbius map is infinitesimal if it is close to the identity map which has $a = d = 1$, $b = c = 0$ and so we require $a = 1+\alpha$, $d = 1+\delta$, $b = \beta$, $c = \gamma$ and get (ignoring second order terms)

$$\delta z = \frac{az+b}{cz+d} - z = \frac{z + \alpha z + \beta}{1 + \delta + \gamma z} - z \approx (z + \alpha z + \beta)(1 - \delta - \gamma z) - z \approx \beta + (\alpha - \delta)z + (-\gamma)z^2 = A + Bz + Cz^2\ ,$$

as required

3.4 Quasiprimary state

Let $L_0|\psi\rangle = h|\psi\rangle$ and $L_1|\psi\rangle = 0$.

Now consider $|\chi\rangle = (L_{-2} - (3/(4h+2))L_{-1}L_{-1})|\psi\rangle$.

Firstly, we can prove $L_0|\chi\rangle = (h+2)|\chi\rangle$

$$\begin{aligned}
L_0 L_{-2}|\psi\rangle &= [L_0, L_{-2}]|\psi\rangle + L_{-2}L_0|\psi\rangle = 2L_{-2}|\psi\rangle + hL_{-2}|\psi\rangle \\
&= (h+2)L_{-2}|\psi\rangle \\
L_0 L_{-1}L_{-1}|\psi\rangle &= [L_0, L_{-1}]L_{-1}|\psi\rangle + L_{-1}[L_0, L_{-1}]|\psi\rangle + L_{-1}L_{-1}L_0|\psi\rangle \\
&= L_{-1}L_{-1}|\psi\rangle + L_{-1}L_{-1}|\psi\rangle + hL_{-1}L_{-1}|\psi\rangle \\
&= (h+2)L_{-1}L_{-1}|\psi\rangle \\
\Rightarrow\ L_0|\chi\rangle &= (h+2)|\chi\rangle\ .
\end{aligned}$$

Now we consider $L_1|\chi\rangle$. We have

$$\begin{aligned}
L_1 L_{-2}|\psi\rangle &= [L_1, L_{-2}]|\psi\rangle + L_{-2}L_1|\psi\rangle = 3L_{-2}|\psi\rangle + 0 \\
&= 3L_{-2}|\psi\rangle \\
L_1 L_{-1}L_{-1}|\psi\rangle &= [L_1, L_{-1}]L_{-2}|\psi\rangle + L_{-1}[L_1, L_{-1}]|\psi\rangle + L_{-1}L_{-1}L_1|\psi\rangle \\
&= 2L_0L_{-1}|\psi\rangle + 2L_{-1}L_0|\psi\rangle + 0 \\
&= 2[L_0, L_{-1}]|\psi\rangle + 4L_{-1}L_0|\psi\rangle \\
&= 2L_{-1}|\psi\rangle + 4hL_{-1}|\psi\rangle \\
&= (4h+2)L_{-1}|\psi\rangle \\
\Rightarrow L_1(L_{-2} - \frac{3}{4h+2}L_{-1}L_{-1})|\psi\rangle &= 3L_{-1}|\psi\rangle - (4h+2)\frac{3}{4h+2}L_{-1}|\psi\rangle \\
&= 0\ .
\end{aligned}$$

3.5 One–, two– and three–point functions

(a) We consider $\langle 0|\phi(z)|0\rangle$ and use $\langle 0|L_{-1} = L_{-1}|0\rangle = 0$

$\langle 0|L_0 = L_0|0\rangle = 0$

(i) $\langle 0|L_{-1} = 0$

$$\Rightarrow \quad 0 = \langle 0|L_{-1}\phi(z)|0\rangle$$

$$= \langle 0|[L_{-1},\phi(z)]|0\rangle + \langle 0|\phi(z)\cancel{L_{-1}|0\rangle}^{\to 0}$$

$$= \frac{\partial}{\partial z}\left(\langle 0|\phi(z)|0\rangle\right)$$

$$\Rightarrow \quad \langle 0|\phi(z)|0\rangle = \text{const} = c.$$

(ii) $\langle 0|L_0 = 0$

$$\Rightarrow \quad 0 = \langle 0|L_0\phi(z)|0\rangle = \langle 0|[L_0,\phi(z)]|0\rangle + \langle 0|\phi(z)\cancel{L_0|0\rangle}^{\to 0}$$

$$= \left(h + z\frac{\partial}{\partial z}\right)\langle 0|\phi(z)|0\rangle$$

$$= h\,\langle 0|\phi(z)|0\rangle \qquad \text{since } \frac{\partial}{\partial z}\langle 0|\phi(z)|0\rangle = 0$$

$$\Rightarrow \quad h = 0 \quad \text{or} \quad \langle 0|\phi(z)|0\rangle = 0$$

$$\Rightarrow \quad \underline{\langle 0|\phi(z)|0\rangle = 0 \quad \text{if } h \neq 0.}$$

(b) Consider $\langle 0| \phi_h(z) \phi_{h'}(w) |0\rangle$

(i) $0 = \langle 0| L_{-1} \phi_h(z) \phi_{h'}(w) |0\rangle$

$$= \langle 0| [L_{-1}, \phi_h(z)] \phi_{h'}(w) |0\rangle$$
$$+ \langle 0| \phi_h(z) [L_{-1}, \phi_{h'}(w)] |0\rangle$$
$$+ \langle 0| \phi_h(z) \phi_{h'}(w) \cancel{L_{-1}|0\rangle} \to 0$$

$$= \langle 0| \frac{\partial}{\partial z}\phi_h(z) \phi_{h'}(w) |0\rangle$$
$$+ \langle 0| \phi_h(z) \frac{\partial}{\partial w} \phi_{h'}(w) |0\rangle$$

$$= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right) \langle 0| \phi_h(z) \phi_{h'}(w) |0\rangle$$

If we put $\xi = z + w$, $\eta = z - w$, then $z = \frac{\xi + \eta}{2}$, $w = \frac{\xi - \eta}{2}$

$$\frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right)$$

So if $\langle 0| \phi_h(z) \phi_{h'}(w) |0\rangle = F(\xi, \eta)$

we have found $\frac{\partial}{\partial \xi} F(\xi, \eta) = 0 \Rightarrow F(\xi, \eta) = f(\eta) = f(z - w)$

i.e. $\langle 0| \phi_h(z) \phi_{h'}(w) |0\rangle = f(z - w)$.

(ii)

$$0 = \langle 0| L_0 \, \phi_h(z) \phi_{h'}(w) |0\rangle$$

$$= \langle 0| [L_0, \phi_h(z)] \phi_{h'}(w) |0\rangle$$

$$+ \langle 0| \phi_h(z) [L_0, \phi_{h'}(w)] |0\rangle \quad + \langle 0| \phi_h(z) \phi_{h'}(w) \cancel{L_0 |0\rangle} \rightarrow 0$$

$$= \langle 0| (h + z\tfrac{\partial}{\partial z}) \phi_h(z) \; \phi_{h'}(w) |0\rangle + \langle 0| \phi_h(z) (h' + w\tfrac{\partial}{\partial w}) \phi_{h'}(w) |0\rangle$$

$$= (h + h' + z\tfrac{\partial}{\partial z} + w\tfrac{\partial}{\partial w}) \langle 0| \phi_h(z) \phi_{h'}(w) |0\rangle$$

$$= (h + h' + z\tfrac{\partial}{\partial z} + w\tfrac{\partial}{\partial w}) \; f(z-w)$$

$$= (h+h') f + z f' - w f'$$

$$= (h+h') f + (z-w) f'(z-w) = 0$$

$$\Rightarrow \quad (h+h') f(t) + t f'(t) = 0$$

$$\Rightarrow \quad \frac{f'(t)}{f(t)} = - \frac{(h+h')}{t}$$

$$\Rightarrow \quad \int \frac{df}{f} = -(h+h') \int \frac{dt}{t}$$

$$\Rightarrow \quad \ln f = \text{const} - (h+h') \ln t$$

$$\Rightarrow \quad f = \text{const.} \; t^{-(h+h')}$$

or $\langle 0| \phi_h(z) \phi_{h'}(w) |0\rangle = \text{const.} \, (z-w)^{-h-h'}$

(iii)

$\langle 0| L_1 = 0$

$$\Rightarrow 0 = \langle 0| L_1 \, \phi_h(z) \, \phi_{h'}(w) |0\rangle$$

$$= \langle 0| [L_1, \phi_h(z)] \, \phi_{h'}(w) |0\rangle$$

$$+ \langle 0| \phi_h(z) [L_1, \phi_{h'}(w)] |0\rangle + \langle 0| \phi_h(z) \phi_{h'}(w) \cancel{L_1 |0\rangle}^{\to 0}$$

$$= \left(2hz + z^2 \frac{\partial}{\partial z} + 2h'w + w^2 \frac{\partial}{\partial w}\right) f(z-w)$$

$$= \left(2hz + z^2 \frac{\partial}{\partial z} + 2h'w + w^2 \frac{\partial}{\partial w}\right) c\,(z-w)^{-h-h'}$$

(c ↑ const.)

$$= \text{const.} \Big[(2hz + 2h'w)(z-w)^{-h-h'} + (-h-h')\, z^2 (z-w)^{-h-h'-1} + w^2 (-h-h')(-1)(z-w)^{-h-h'-1} \Big]$$

$$= \frac{\text{const.}}{(z-w)^{h+h'+1}} \Big[(2hz + 2h'w)(z-w) - (h+h')(z^2 - w^2) \Big]$$

$$= \frac{\text{const.}}{(z-w)^{h+h'+1}} \Big[(h-h')(z^2 - 2zw + w^2) \Big]$$

$$= (h-h') \cdot \text{const.} \, (z-w)^{-h-h'+1} = 0$$

$$\Rightarrow \begin{cases} \langle 0| \phi_h(z) \, \phi_{h'}(w) |0\rangle = 0 \quad \text{~~unless~~ if } h \neq h' \\ \langle 0| \phi_h(z) \, \phi_h(w) |0\rangle = \dfrac{\text{const.}}{(z-w)^{2h}} \end{cases}$$

4. Full infinite symmetry

4.1 Conformal invariance in light-cone coordinates

Light-cone coordinates for Minkowski space are defined as $x^+ = t + x, x^- = t - x$.

(a) The simplest way to find the metric is from the line element,

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}^2x^2 = \left(\frac{\mathrm{d}x^+ + \mathrm{d}x^-}{2}\right)^2 - \left(\frac{\mathrm{d}x^+ - \mathrm{d}x^-}{2}\right)^2 \\ &= \mathrm{d}x^+\mathrm{d}x^- = \eta_{++}\mathrm{d}x^+\mathrm{d}x^+ + 2\eta_{+-}\mathrm{d}x^+\mathrm{d}x^- + \eta_{--}\mathrm{d}x^-\mathrm{d}x^- \ , \end{aligned}$$

from which we can read off that $\eta_{++} = \eta_{--} = 0, \eta_{+-} = \eta_{-+} = \frac{1}{2}$ so that

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \ , \ \eta^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \ .$$

(b) We first find

$$T^\mu{}_\mu = T^+{}_+ + T^-{}_- = \eta^{+-}T_{-+} + \eta^{-+}T_{+-} = 4T_{+-} = 0 \ .$$

Next,

$$\partial^\nu T_{\nu+} = \eta^{\mu\nu}\partial_\mu T_{\nu+} = 2\partial_+ T_{-+} + 2\partial_- T_{++} = 2\partial_- T_{++} = 0 \ ,$$

and finally

$$\partial^\nu T_{\nu-} = \eta^{\mu\nu}\partial_\mu T_{\nu-} = 2\partial_+ T_{--} + 2\partial_- T_{+-} = 2\partial_+ T_{--} = 0 \ ,$$

(c) Finally, we use the chain rule

$$\frac{\partial}{\partial t} = \frac{\partial x^+}{\partial t}\partial_+ + \frac{\partial x^-}{\partial t}\partial_- = \partial_+ + \partial_- \ , \frac{\partial}{\partial x} = \frac{\partial x^+}{\partial x}\partial_+ + \frac{\partial x^-}{\partial x}\partial_- = \partial_+ - \partial_- \ ,$$

to re-write

$$\begin{aligned} & \frac{d}{dt}\int_{-\infty}^{\infty} \left(f(x^+)T_{++} + g(x^-)T_{--}\right)\,\mathrm{d}x \\ = & \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(f(x^+)T_{++} + g(x^-)T_{--}\right)\,\mathrm{d}x \\ = & \int_{-\infty}^{\infty} (\partial_+ + \partial_-)\left(f(x^+)T_{++} + g(x^-)T_{--}\right)\,\mathrm{d}x \\ = & \int_{-\infty}^{\infty} \left[\partial_+(f(x^+)T_{++}) + \partial_-(g(x^-)T_{--})\right]\,\mathrm{d}x \\ = & \int_{-\infty}^{\infty} \left[(\partial_+ - \partial_-)(f(x^+)T_{++}) + (\partial_- - \partial_+)(g(x^-)T_{--})\right]\,\mathrm{d}x \\ = & \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left[f(x^+)T_{++} - g(x^-)T_{--}\right]\,\mathrm{d}x \\ = & \ 0 \end{aligned}$$

where we repeatedly used

$$\partial_-(f(x^+)T_{++}) = 0 \ , \ \ \partial_+(f(x^-)T_{--}) = 0 \ .$$

4.2 Highest weight states

To show that $L_1\psi = 0$ and $L_2\psi = 0$ implies $L_m|\psi\rangle = 0$ for all $m > 0$.

We have $[L_m, L_1] = (m-1)L_{m+1}$, or

$$L_m = \frac{1}{m-2}[L_{m-1}, L_1] \;\; m > 2 \; ,$$

and hence

$$L_m|\psi\rangle = \frac{1}{n-2}\left(L_{m-1}L_1|\psi\rangle - L_1 L_{m-1}|\psi\rangle\right) \; .$$

This allows us to prove the result by induction.

Suppose that $L_p|\psi\rangle = 0$ for $p = 1, \ldots, m-1$ with $m \geq 3$, then $L_m|\psi\rangle = \frac{1}{m-2}\left(L_{m-1}L_1|\psi\rangle - L_1 L_{m-1}|\psi\rangle\right) = 0$

However, by assumption $L_1|\psi\rangle = L_2|\psi\rangle = 0$.

Hence $L_p|\psi\rangle = 0$ for all $p \geq 1$.

5. Brute Force

5.1 Determinant

This was not asekd for, but just for the record thuis is how to work out the matrix M_2.
Let $|h\rangle$ be a primary state of weight h and we normalised it as $\langle h||h\rangle = 1$.
The space of descendants at level 2 is two dimensional with basis states

$$L_{-2}|h\rangle \ , \quad L_{-1}L_{-1}|h\rangle \ ,$$

We can calculate their overlaps,

$$\begin{aligned}
\langle h|L_2L_{-2}|h\rangle &= \langle h|L_{-2}L_2 + [L_2, L_{-2}]|h\rangle \\
&= \langle h|(4L_0 + \frac{c}{2})|h\rangle \qquad\qquad (\text{using } L_2|h\rangle = 0) \\
&= (4h + \frac{c}{2})\langle h|h\rangle = (4h + \frac{c}{2}) \\
\langle h|L_2L_{-1}L_{-1}|h\rangle &= \langle h|L_{-1}L_{-1}L_2 + [L_2, L_{-1}]L_{-1} + L_{-1}[L_2, L_{-1}]|h\rangle \\
&= \langle h|0 + 3L_1L_{-1} + L_{-1}(3L_1)|h\rangle \\
&= \langle h|3L_{-1}L_1 + 3[L_1, L_{-1}] + 0|h\rangle \\
&= \langle h|0 + 6L_0|h\rangle = 6h \\
\langle h|L_1L_1L_{-2}|h\rangle &= (\langle h|L_2L_{-1}L_{-1}|h\rangle)^\dagger \ = \ 6h \\
\langle h|L_1L_1L_{-1}L_{-1}|h\rangle &= \langle h|L_1(L_{-1}L_{-1}L_1 + [L_1, L_{-1}]L_{-1} + L_{-1}[L_1, L_{-1}])|h\rangle \\
&= \langle h|L_1(0 + 2L_0L_{-1} + L_{-1}(2L_0))|h\rangle \\
&= \langle h|L_1(2(h+1)L_{-1} + 2hL_{-1})|h\rangle = (4h+2)\langle h|L_1L_{-1}|h\rangle \\
&= (4h+2)\langle h|L_{-1}L_1 + 2L_0|h\rangle = 2h(4h+2) = 4h(2h+1)
\end{aligned}$$

This means the matrix of inner products is

$$M_2 = \det\begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + c/2 \end{pmatrix} \ = \ 4h(2h+1)(4h+c/2) - 36h^2 = 32h(h^2 + hc + c/16 - 5h/8)$$

Now we calculate:

$$\begin{aligned}
(h - h_{12})(h - h_{21}) &= \left(h - \left[\frac{3}{4t} - \frac{1}{2}\right]\right)\left(h - \left[\frac{3t}{4} - \frac{1}{2}\right]\right) \\
&= h^2 + h\left(1 - \frac{3t}{4} - \frac{3}{4t}\right) + \frac{13}{16} - \frac{3t}{8} - \frac{3}{8t} \\
&= h^2 + h\left(\frac{13 - 6t - 6/t}{8} - \frac{5}{8}\right) + \frac{13 - 6t - 6/t}{16} \\
&= h^2 + h(c/8 - 5/8) + c/16 \ .
\end{aligned}$$

where we used $c = 13 - 6t - 6/t$. So, in total, since $h_{11} = 0$,

$$\det(M_2) = 32\,(h - h_{11})\,(h - h_{12})\,(h - h_{21}) \ .$$

We have

$$c = 13 - 6t - \frac{6}{t} \qquad \Rightarrow t = -\frac{13 + c \pm \sqrt{(1-c)(25-c)}}{12} \ .$$

This means that t is complex if $1 < c < 15$, in fact it will be a pure phase, $t = \exp(i\theta)$ so that

$$c = 13 - 12\cos\theta \ .$$

Correspondingly, the only real values of h_{rs} will be when t appears symmetrically. We have

$$h_{rs} = \frac{r^2-1}{4t} + \frac{(s^2-1)t}{4} - \frac{rs-1}{2} = \frac{r^2+s^2-2}{2}\cos\theta + \frac{r^2-s^2}{2i}\sin\theta - \frac{rs-1}{2}$$

This is only real for positive r and s if $r = s$ and so the only vanishing curves in the region $1 < c < 25$ are

$$h = h_{rr} = \frac{r^2-1}{4}\left(t + \frac{1}{t}\right) - \frac{r^2-1}{2} = \frac{r^2-1}{2}(\cos\theta - 1) = \frac{r^2-1}{24}\ (1-c)$$

which are straight lines.

6. Differential equations

6.1 Singular vectors

We have to find the conditions under which $|\phi\rangle = (L_{-2}L_{-2} - (3/5)L_{-4})|0\rangle$ is a highest weight state. We check

$$\begin{aligned}
L_1|\phi\rangle &= L_1(L_{-2}L_{-2} - \frac{3}{5}L_{-4})|0\rangle \\
&= (L_{-2}L_{-2}L_1 + [L_1, L_{-2}]L_{-2} + L_{-2}[L_1, L_{-2}] - \frac{3}{5}L_{-4}L_{-1} - \frac{3}{5}[L_1, L_{-4}])|0\rangle \\
&= (0 + 3L_{-1}L_{-2} + 3L_{-2}L_{-1} - 0 - \frac{3}{5}(5L_{-3}))|0\rangle \\
&= (3L_{-2}L_{-1} + 3[L_{-1}, L_{-2}] + 0 - 3L_{-3})|0\rangle \\
&= (0 + 3L_{-1} - 3L_{-1})|0\rangle \\
&= 0
\end{aligned}$$

and so we see that $|\phi\rangle$ is always a quasi-primary state.

We only have now to check

$$\begin{aligned}
L_2|\phi\rangle &= L_2(L_{-2}L_{-2} - \frac{3}{5}L_{-4})|0\rangle \\
&= (L_{-2}L_{-2}L_2 + [L_2, L_{-2}]L_{-2} + L_{-2}[L_2, L_{-2}] - \frac{3}{5}L_{-4}L_2 - \frac{3}{5}[L_2, L_{-4}])|0\rangle \\
&= (0 + (4L_0 + \frac{c}{2})L_{-2} + L_{-2}(4L_0 + \frac{c}{2}) - 0 - \frac{18}{5}L_{-2})|0\rangle \\
&= (8 + c - \frac{18}{5})L_{-2}|0\rangle \\
&= (\frac{22}{5} + c)L_{-2}|0\rangle
\end{aligned}$$

so that $|\phi\rangle$ is a highest weight state when $c = -22/5$.

This is the central charge of the Lee-Yang model, the minimal model with $t = 2/5$.

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7. Recursion relations

7.1 For discussion in the live lecture.

8. AGT: an exact formula

8.1 Liouville theory

The standard parametrisation of c and h is

$$c = 1 + 6Q^2\ ,\quad h = \alpha(Q-\alpha)\ ,\quad Q = b + 1/b\ .$$

If b is real then we can put $b = \pm e^u$ and so $Q = \pm 2\cosh u$ and $c = 1 + 24\cosh^2 u \geq 25$.

If b is pure imaginary then we can put $b = \pm i e^v$ and so $Q = \pm 2i\sinh v$ and $c = 1 - 24\sinh^2 v \leq 1$.

If $\alpha = Q/2 + iP$ then $h = \alpha(Q-\alpha) = (Q/2+iP)(Q/2-iP) = Q^2/4 + P^2 = \frac{c-1}{24} + P^2$.

This means that there are no null states in any of the representations which occur in (real-coupling) Liouville theory.

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9. "Old" Conformal Bootstrap

Ising model

We use $\quad |\psi\rangle = \left(L_{-2} - \frac{4}{3}L_{-1}L_{-1}\right)|\sigma\rangle$

and re-write it as: $\quad = \left(L_{-2} - L_0 + \frac{1}{16} - \frac{4}{3}\left(L_{-1} - L_0 + \frac{17}{16}\right)\left(L_{-1} - L_0 + \frac{1}{16}\right)\right)|\sigma\rangle$

$$\Rightarrow \quad \langle\sigma|\sigma(1)|\psi\rangle = \left(\frac{2}{16} - \frac{4}{3}\left(\frac{17}{16}\right)\left(\frac{1}{16}\right)\right)\langle\sigma|\sigma(1)|\sigma\rangle$$

$$= \frac{7}{196}\langle\sigma|\sigma(1)|\sigma\rangle = 0$$

$$\Rightarrow \langle\sigma|\sigma(1)|\sigma\rangle = 0.$$

Now $\quad \langle h|\sigma(1)|\psi\rangle = \left[\left(\frac{3}{16} - h\right) - \frac{4}{3}\left(\frac{18}{16} - h\right)\left(\frac{2}{16} - h\right)\right]\langle h|\sigma(1)|\sigma\rangle$

$$= \left[-\frac{4}{3}h^2 + h\left(\frac{1}{6} + \frac{9}{6} - 1\right) + \left(\frac{3}{16} - \frac{4}{3}\cdot\frac{18}{16}\right)\right]\langle h|\sigma(1)|\sigma\rangle$$

$$= -\frac{4}{3}h\left(h - \frac{1}{2}\right)\langle h|\sigma(1)|\sigma\rangle$$

This can only be non zero if $h=0$ or $h = 1/2$.

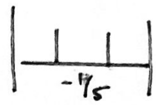
These are the two other entries in the 'Kac table' for $c = 1/2$

h_{rs} (r ↑, s →)

1/2	1/16	0
0	1/16	1/2

Lee-Yang

Here we calculate the structure constant $C_{\phi\phi\phi}$ in the Lee-Yang model.

$$\langle \phi | \phi \phi | \phi \rangle = (C_{\phi\phi 0})^2 \Big|$$  $$\Big|^2 + (C_{\phi\phi\phi})^2 \Big|$$  $$\Big|^2$$

$$= (C_{\phi\phi 0})^2 \, |z(1-z)|^{4/5} \, |F(3/5, 4/5; 6/5, z)|^2$$
$$+ (C_{\phi\phi\phi})^2 \, |z|^{2/5} \, |1-z|^{4/5} \, |F(3/5, 2/5; 4/5; z)|^2$$

} from 5.6

We can take $z \in \mathbb{R}$ to simplify matters.

This should be equal to

$$(C_{\phi\phi 0})^2 \, |(1-z)z|^{4/5} \, |F(3/5, 4/5; 6/5; 1-z)|^2$$
$$+ (C_{\phi\phi\phi})^2 \, |1-z|^{2/5} \, |z|^{4/5} \, |F(3/5, 2/5; 4/5; 1-z)|^2$$

From 3.5 we have

- $$F(3/5, 4/5; 6/5, 1-z) = \frac{\Gamma(6/5)\Gamma(-1/5)}{\Gamma(3/5)\Gamma(2/5)} F(3/5, 4/5, 6/5, z)$$
$$+ z^{-1/5} \frac{\Gamma(6/5)\Gamma(1/5)}{\Gamma(3/5)\Gamma(4/5)} F(3/5, 2/5, 4/5, z)$$

- $$F(3/5, 2/5, 4/5, 1-z) = \frac{\Gamma(4/5)\Gamma(-1/5)}{\Gamma(1/5)\Gamma(2/5)} F(3/5, 2/5; 6/5, z)$$
$$+ z^{-1/5} \frac{\Gamma(4/5)\Gamma(1/5)}{\Gamma(3/5)\Gamma(2/5)} F(1/5, 2/5; 4/5, z)$$

Need extra identity:

$$F(a, b, c, z) = (1-z)^{c-a-b} F(c-a, c-b, c; z)$$

$$\Rightarrow F(3/5, 2/5; 4/5; 1-z) = (1-z)^{-1/5} \frac{\Gamma(4/5)\Gamma(-1/5)}{\Gamma(1/5)\Gamma(2/5)} F(3/5, 4/5; 6/5; z)$$
$$+ z^{-1/5}(1-z)^{1/5} \frac{\Gamma(4/5)\Gamma(1/5)}{\Gamma(3/5)\Gamma(2/5)} F(3/5, 2/5, 4/5, z)$$

We get (z real $0<z<1$)

$$(C_{\phi\phi 0})^2 z^{4/5}(1-z)^{4/5} F(3/5, 4/5; 6/5; z)^2$$

$$+ (C_{\phi\phi\phi})^2 z^{2/5}(1-z)^{4/5} F(3/5, 2/5; 4/5; z)^2$$

$$= (C_{\phi\phi 0})^2 z^{4/5}(1-z)^{4/5} \left\{ \frac{\Gamma(6/5)\Gamma(-1/5)}{\Gamma(3/5)\Gamma(2/5)} F(3/5, 4/5, 6/5, z) + z^{-1/5} \frac{\Gamma(6/5)\Gamma(1/5)}{\Gamma(3/5)\Gamma(4/5)} F(3/5, 2/5; 4/5, z) \right\}^2$$

$$+ (C_{\phi\phi\phi})^2 z^{4/5}(1-z)^{2/5} \left\{ (1-z)^{1/5} \frac{\Gamma(4/5)\Gamma(-1/5)}{\Gamma(1/5)\Gamma(2/5)} F(3/5, 4/5; 6/5; z) + z^{-1/5}(1-z)^{-1/5} \frac{\Gamma(4/5)\Gamma(1/5)}{\Gamma(3/5)\Gamma(2/5)} F(3/5, 2/5; 4/5; z) \right\}^2$$

Comparing coefficients:

(i) $z^{4/5}(1-z)^{4/5} F(3/5, 4/5; 6/5; z)^2$: $\quad 1 = \left(\frac{\Gamma(6/5)\Gamma(-1/5)}{\Gamma(3/5)\Gamma(2/5)}\right)^2 + \left(\frac{C_{\phi\phi\phi}}{C_{\phi\phi 0}}\right)^2 \left(\frac{\Gamma(4/5)\Gamma(-1/5)}{\Gamma(1/5)\Gamma(2/5)}\right)^2$

(ii) $z^{2/5}(1-z)^{4/5} F(3/5, 2/5; 4/5; z)^2$: $\quad 1 = \left(\frac{\Gamma(4/5)\Gamma(1/5)}{\Gamma(3/5)\Gamma(2/5)}\right)^2 + \left(\frac{C_{\phi\phi 0}}{C_{\phi\phi\phi}}\right)^2 \left(\frac{\Gamma(6/5)\Gamma(1/5)}{\Gamma(4/5)\Gamma(3/5)}\right)^2$

(iii)

$z^{3/5}(1-z)^{4/5} F(3/5, 4/5; 6/5; z) \cdot F(3/5, 2/5; 4/5; z)$

$$0 = \frac{\Gamma(6/5)^2\Gamma(-1/5)\Gamma(1/5)}{\Gamma(3/5)^2\Gamma(2/5)\Gamma(4/5)} + \frac{\Gamma(4/5)^2\Gamma(-1/5)\Gamma(1/5)}{\Gamma(2/5)^2\Gamma(1/5)\Gamma(3/5)} \left(\frac{C_{\phi\phi\phi}}{C_{\phi\phi 0}}\right)^2$$

n.b. identities: $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$, $\quad \Gamma(x+1) = x\Gamma(x)$

If $C_{\phi\phi 0} = 1$, natural: $\langle \phi(z)\phi(0)\rangle = |z|^{4/5}$

Then

$$(C_{\phi\phi\phi})^2 = -\frac{\Gamma(6/5)^2\,\cancel{\Gamma(-1/5)\Gamma(1/5)}}{\Gamma(3/5)^2\,\Gamma(2/5)\Gamma(4/5)}\cdot\frac{\Gamma(2/5)^2\,\Gamma(1/5)\Gamma(3/5)}{\Gamma(4/5)^2\,\cancel{\Gamma(-1/5)\Gamma(1/5)}}$$

$$= -\frac{\Gamma(6/5)^2\,\Gamma(2/5)\,\Gamma(1/5)}{\Gamma(3/5)\,\Gamma(4/5)^3}$$

$$= -\frac{1}{25}\,\frac{\Gamma(2/5)\,\Gamma(1/5)^3}{\Gamma(3/5)\,\Gamma(4/5)^3}.$$

This solves the other eqns as well.

Note that $(C_{\phi\phi\phi})^2 < 0$

Solution: $C_{\phi\phi\phi}$ is purely imaginary

Better solution: We cannot normalise all fields to +1 and get real constants. So instead have a metric, $\langle\phi_i\phi_j\rangle = \dfrac{g_{ij}}{|z-w|^{2\Delta_i}}$

$$\langle\phi|\phi\,\phi|\phi\rangle = (C_{\phi\phi}^{\phi})^2 g_{\phi\phi}\left|\mathcal{F}_{-1/5}\right|^2 + (g_{\phi\phi})^2\left|\mathcal{F}_{0}\right|^2$$

Take $g_{\phi\phi} = -1$, $\quad (C_{\phi\phi}^{\phi})^2 = +\dfrac{1}{25}\,\dfrac{\Gamma(2/5)\Gamma(1/5)^3}{\Gamma(3/5)\Gamma(4/5)^3}$

Can get real structure constants at cost of negative norm of $\langle\phi|\phi\rangle$

But the space of states $L_{-m}^{n_m}\cdots L_{-1}^{n_1}|\phi\rangle$ has an indefinite inner product so this is a small cost.

10. “New” Conformal Bootstrap

For discussion in the live lecture.