Conformal Blocks: some more exercises

Here are some more exercises to test thinking and understanding, expand on the lecture as well as some standard standard technical exercises, organised more or less by section.

2. Generalities on conformal transformations

2.4 The Möbius group

Consider two dimensional Euclidean space in complex coordinates $(z, \bar{z})$. We shall ignore $\bar{z}$ dependence from now on.

Consider a four-point function

$$\langle \varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3)\varphi_4(z_4)\rangle \ .$$

Find the Möbius map

$$z \mapsto \frac{az+b}{cz+d} \ ,$$

which sends

$$z_1 \mapsto \infty \ , \ \ z_2 \mapsto 1 \ , \ \ z_3 \mapsto \zeta \ , \ \ z_4 \mapsto 0 \ .$$

What is the value of ζ?

Show that ζ is invariant under

$$z_i \mapsto \frac{Az_i + B}{Cz_i + D} \ .$$

Consider permutations of z_i. Show that ζ is mapped to one of

$$\zeta \ , \ \ 1-\zeta \ , \ \ \frac{1}{\zeta} \ , \ \ \frac{1}{1-\zeta} \ , \ \ \frac{\zeta-1}{\zeta} \ , \ \ \frac{\zeta}{\zeta-1} \ .$$

Hint: first find which permutations leave ζ invariant.

2.5 Cross-ratios in arbitrary dimension

In $d \geq 2$ dimensions, global conformal transformations allow one to map any four points $\boldsymbol{r}_i$ to

$$\boldsymbol{r}_1 \mapsto \infty \ , \ \ \boldsymbol{r}_2 \mapsto (1,0,0,..) \ , \ \ \boldsymbol{r}_3 \mapsto (x,y,0,..) \ , \ \ \boldsymbol{r}_1 \mapsto (0,0,0,..) \ .$$

The standard definition of the cross-ratios u and v is

$$u = \frac{r_{12}\, r_{34}}{r_{13}\, r_{24}} \ , \ \ v = \frac{r_{14}\, r_{23}}{r_{13}\, r_{24}} \ , \quad \text{where } r_{ij} = (\boldsymbol{r}_i - \boldsymbol{r}_j)^2 \ .$$

Verify that if $z = x + iy$ then

$$u = z\,\bar{z} \ , \ \ v = (1-z)(1-\bar{z}) \ .$$

Verify that in two dimensions these agree with the standard definition of the cross-ratio ζ as in question 2.4.

2.6 Casimir operator

The "standard" route to find the global blocks is to use th Casimir operator for the conformal group. In two dimensions, because the lie algebra is $sl(2) \oplus sl(2)$, there are two independent Casimirs.

(a) Using the commutation relations valid for $m \in \{-1, 0, 2\}$,

$$[L_m, L_n] = (m-n)L_{m+n} \ ,$$

show that the operator

$$C_2 = L_0 L_0 - L_1 L_{-1} - L_{-1} L_1$$

commutes with the algebra generators.

It is easier to use this in the form

$$C_2' = L_0 L_0 - 2(L_{-1} - L_0)L_1 - L_0(L_1 + 1) \ .$$

Verify that $C_2 = C_2'$.

(b) Verify that on a quasiprimary state $|h\rangle$,

$$C_2|h\rangle = (h^2 - h)|h\rangle \ .$$

(c) Inserting C_2 in the intermediate channel, projecting onto the quasiprimary representation with weight h and using appropriate commutation relations, show that

$$\left[(z\partial + h_3 + h_4)(z\partial + h_3 + h_4) - (z^2\partial + 2h_3 z)(h_2 - h_1 + z\partial + h_3 + h_4) - (h^2 - h)\right] F(z) = 0 \ .$$

(d) Show that $G(z)$ defined through $F = z^{h-h_3-h_4}G$ satisfies the hypergeometric equation

$$z(z-1)G'' + ((2h + 1 - h_{12} + h_{34})z - 2h)G' + (h - h_{12})(h + h_{34})G = 0 \ .$$

(e) Check that the result derived in the lectures is one of the two solutions of this equation.

7. Recursion relations

7.2

Write out the recursion steps needed to calculate the conformal block to order z^4.

How many recursive steps are required to calculate the conformal block to order z^n?

8. AGT: an exact formula

8.2

Write out the pairs of Young tableaux needed to calculate the conformal block to order z^4. Recall that the sum is over pairs of Young tableaux weighted by $z^{(\text{total number of boxes})}$

How many calculational steps are required to calculate the conformal block to order z^n?

How does this compare with the result of 8.2?