
Conformal Blocks: notes, references and exercises

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1. Introduction

2. Generalities on conformal transformations

A nice of lecture notes on CFT in $d > 2$:

Lectures on Conformal Field Theories in more than two dimensions
Hugh Osborn (2019)

The standard early references on conformal partial waves are from the 1970s by the group of S. Ferrara, R. Gatto, G. Parisi, A.F. Grillo:

Covariant expansion of the conformal four-point function
Nucl.Phys.B 49 (1972) 77-98, Nucl.Phys.B 53 (1973) 643-643 (erratum)

Analyticity properties and asymptotic expansions of conformal covariant green's functions
Nuovo Cim.A 19 (1974) 667-695

Properties of Partial Wave Amplitudes in Conformal Invariant Field Theories
Nuovo Cim.A 26 (1975) 226

The specialisation to $d = 2$ in the 80's mean that relatively little attention seemed to be paid to the general case until AdS/CFT at the end of the 90's. Some referenceas from the early 2000s:

hep-th/0011040
Conformal Four Point Functions and the Operator Product Expansion
F.A. Dolan, H. Osborn

arxiv.org/hep-th/0309180
Conformal Partial Waves and the Operator Product Expansion
F.A. Dolan, H. Osborn

Latest work includes

arXiv:2001.08778
Distributions in CFT I. Cross-Ratio Space
Petr Kravchuk, Jiaxin Qiao, Slava Rychkov

3. Specialisation to $d = 2$

The standard text book, the "Big Yellow Book"

[YBK] Conformal Field Theory
P. Di Francesco, P. Mathieu and D. Sénéchal, 1997

4. Full infinite symmetry

The seminal paper introducing many ideas in 2d CFT:

[BPZ] Nuclear Physics B (241) 1984, 333-380
Infinite conformal symmetry in two-dimensional quantum field theory
A A Belavin, A M Polyakov and A B Zamolodchikov

For a recent discussion of the crossing and the phases that occur and why:
arxiv.org/2001.05055
Fermionic CFTs and classifying algebras
Ingo Runkel, Gerard M. T. Watts

5. Brute Force

No specific references, the method idea is immedate.

6. Differential equations

This method was introduced (along with many other ideas) in [BPZ]

7. Recursion relations

They were introduced in

Theoretical and Mathematical Physics volume 73, pages1088–1093 (1987)
Conformal symmetry in two-dimensional space: Recursion representation of conformal block
Al. B. Zamolodchikov

They are discussed and used in

hep-th/9506136
Structure Constants and Conformal Bootstrap in Liouville Field Theory
A.B.Zamolodchikov, Al.B.Zamolodchikov

hep-th/0107118
A non-rational CFT with $c = 1$ as a limit of minimal models
I. Runkel, G. M. T. Watts

They were resummed into a closed expression (with a few typos) in

arxiv.org/1502.07742
Virasoro conformal blocks in closed form
Eric Perlmutter

8. AGT: an exact formula

The AGT (Alday, Gaiotto, Tachikawa) correspondence was announced in:

arxiv.org/0906.3219
Liouville Correlation Functions from Four-dimensional Gauge Theories
Luis F. Alday, Davide Gaiotto, Yuji Tachikawa

Some discussion of the AGT and its relation to the states:
arxiv.org/0912.0504
On AGT conjecture

V.A. Fateev, A.V. Litvinov

arxiv.org/1012.1312
On combinatorial expansion of the conformal blocks arising from AGT conjecture
V.A. Alba, V.A. Fateev, A.V. Litvinov, G.M. Tarnopolsky

9. "Old" Conformal Bootstrap

Again, see [BPZ], [YBK].

10. "New" Conformal Bootstrap

This was started in

arxiv.org/0807.0004
Bounding scalar operator dimensions in 4D CFT
Riccardo Rattazzi, Vyacheslav S. Rychkov, Erik Tonni, Alessandro Vichi

One of its most striking results in for the $3d$ Ising model:

arxiv.org/1203.6064
Solving the 3D Ising Model with the Conformal Bootstrap
Sheer El-Showk, Miguel F. Paulos, David Poland, Slava Rychkov, David Simmons-Duffin, Alessandro Vichi

A good introduction is

arxiv.org/1602.07982
TASI Lectures on the Conformal Bootstrap
David Simmons-Duffin

The example on an application of the conformal bootstrap came from these lectures:

arxiv.org/1601.05000
EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions
Slava Rychkov

11. New and other directions

Applications to the eigenstate thermalisation hypothesis:
arxiv.org/1501.05315
Virasoro Conformal Blocks and Thermality from Classical Background Fields
A. Liam Fitzpatrick, Jared Kaplan, Matthew T. Walters

A different approach to the large c limit: arxiv.org/1109.6764
The large central charge limit of conformal blocks
Vladimir Fateev, Sylvain Ribault

Exercises

Here are some exercises to test thinking and understanding, expand on the lecture as well as some standard standard technical exercises, organised more or less by section.

2. Generalities on conformal transformations

2.1 Special conformal transformations

Show that the general quadratic solution $\epsilon_\mu = \gamma_{\mu\nu\rho}x^\nu x^\rho$ to the equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \left(\frac{2}{d}\right) \delta_{\mu\nu}(\delta^{\sigma\tau}\partial_\sigma \epsilon_\tau) ,$$

in flat d-dimensional Euclidean space is $\epsilon^\mu = 2x^\mu(b \cdot x) - b^\mu x^2$, where b^μ are constants. nb you can assume $\gamma_{\mu\nu\rho} = \gamma_{\mu\rho\nu}$.

Consider the coordinate transformation of flat space

$$x'^\mu = \frac{x^\mu - (x^2)b^\mu}{1 - 2\,\mathbf{x}\cdot\mathbf{b} + x^2 b^2} , \tag{1}$$

where $\mathbf{b}$ is a constant vector. Show that $\frac{x'^\mu}{(x')^2} = \frac{x^\mu}{x^2} - b^\mu$ .Hint: find $(x')^2$ first.

Consider now the vector $y^\mu(t)$ with real parameter t defined by

$$y(t)^\mu = \frac{x^\mu - (x^2)te^\mu}{1 - 2t\,\mathbf{x}\cdot\mathbf{e} + x^2 t^2} ,$$

where $\mathbf{e} = \hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$. We denote $|\mathbf{b}| = b$ so that $\mathbf{b} = b\,\mathbf{e}$. Show that

$$(a)\;\; \mathbf{e}\cdot\mathbf{y} = \frac{\mathbf{e}\cdot\mathbf{x} - tx^2}{1 - 2t\,\mathbf{x}\cdot\mathbf{e} + x^2t^2} \;,(b)\;\; y^2 = \frac{x^2}{1 - 2t\,\mathbf{x}\cdot\mathbf{e} + x^2t^2} \;,(c)\;\; \frac{\mathrm{d}y^\mu}{\mathrm{d}t} = 2(\mathbf{e}\cdot\mathbf{y})y^\mu - y^2\,e^\mu \;.$$

This shows that $x^\mu \to y^\mu(t)$ is a finite special conformal transformation. Noting that $\mathbf{x}' = \mathbf{y}(b)$, we see that a finite special conformal transformation takes the form of an inversion (in the unit circle) followed by a translation followed by another inversion in the unit circle.

2.2 Classical scale invariant Lagrangians

Consider a scalar field which transforms under an infinitesimal coordinate transformation $x^\mu \to x'^\mu$ as

$$\phi(x) = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right|^{\Delta/d} \phi(x') ,$$

where $|\partial x'^\mu/\partial x^\nu|$ is the Jacobian of the transformation.

(a) Show that under a scale transformation $x'^\mu = \lambda x^\mu$, the field ϕ has scale dimension Δ.

(b) Show that under an infinitesimal transformation, $\delta x^\mu = \alpha^\mu$, the variation of ϕ is

$$\delta\phi = \frac{\Delta}{d}(\partial_\mu \alpha^\mu)\phi \;+\; \alpha^\sigma \partial_\sigma \phi \;.$$

(c) Show that the infinitesimal variation of ϕ under an infinitesimal scale transformation $\delta x^\mu = \epsilon x^\mu$ is

$$\delta_\epsilon \phi = \epsilon(\Delta\phi + x^\nu \partial_\nu \phi) \ . \qquad (\dagger)$$

(d) Show that the variation of the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\partial_\sigma\phi\partial^\sigma\phi - V(\phi) \ ,$$

where ϕ transforms as ($\dagger$), is a total derivative under an infinitesimal scale transformation provided $\Delta = (d/2) - 1$, and $V = c\,\phi^{D/\Delta}$ for some constant c. [What are these potentials?]

2.3 Scale invariance is not conformal invariance

Consider the following Lagrangian in four dimensions

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_A \ , \quad \text{where} \ \ \mathcal{L}_\phi = \partial_\sigma\bar{\phi}\,\partial^\sigma\phi \ , \quad \mathcal{L}_A = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ ,$$

where ϕ is a complex scalar field, $\bar{\phi}$ is its conjugate and $F_{\mu\nu} = \partial A_\mu - \partial A_\nu$ is the field strength of a gauge field A_μ. Under conformal transformations, the fields vary as

$$\delta\phi = \frac{1}{4}(\partial\cdot\epsilon)\phi + \epsilon^\sigma\partial_\sigma\phi \ , \quad \delta\bar{\phi} = \frac{1}{4}(\partial\cdot\epsilon)\bar{\phi} + \epsilon^\sigma\partial_\sigma\bar{\phi} \ , \quad \delta A_\mu = \epsilon^\sigma\partial_\sigma A_\mu + A_\sigma\partial_\mu\epsilon^\sigma \ .$$

(a) Show that

$$\delta\mathcal{L}_\phi = \partial_\mu\left[\epsilon^\mu(\partial\phi\cdot\partial\bar{\phi}) + \frac{1}{4}\phi\bar{\phi}\partial^\mu(\partial\cdot\epsilon)\right] - \frac{1}{4}\phi\bar{\phi}\partial_\sigma\Box\epsilon^\sigma + \left[\partial_\sigma\epsilon_\tau + \partial_\tau\epsilon_\sigma - \frac{1}{2}(\partial\cdot\epsilon)\eta_{\sigma\tau}\right]\partial^\sigma\phi\,\partial^\tau\bar{\phi} \ ,$$

hence $\mathcal{L}_\phi$ is invariant (up to a total derivative) for conformal transformations (explain why).

(b) Show that

$$\delta F_{\mu\nu} = (\partial_\mu\epsilon^\sigma)F_{\sigma\nu} + (\partial_\nu\epsilon^\sigma)F_{\mu\sigma} + \epsilon^\sigma\partial_\sigma F_{\mu\nu} \ .$$

$$\delta\mathcal{L}_A = \partial_\sigma(-\frac{1}{4}\epsilon^\sigma F_{\mu\nu}F^{\mu\nu}) - \tfrac{1}{2}\left[\partial_\sigma\epsilon_\tau + \partial_\tau\epsilon_\sigma - \frac{1}{2}(\partial\cdot\epsilon)\eta_{\sigma\tau}\right]F^\sigma{}_\nu F^{\tau\nu} \ ,$$

so again $\mathcal{L}_A$ is invariant (up to a total derivative) for conformal transformations.

(c) Now consider the usual interaction term (up to a factor of $-ie$)

$$\mathcal{L}_1 = A^\mu J_\mu \quad \text{where} \quad J_\mu = (\bar{\phi}\,\partial_\mu\phi - \phi\,\partial_\mu\bar{\phi}) \ .$$

Show that

$$\delta J_\mu = \tfrac{1}{2}(\partial\cdot\epsilon)J_\mu + \epsilon^\sigma\partial_\sigma J_\mu + J_\tau\partial_\mu\epsilon^\tau \ .$$

$$\delta\mathcal{L}_1 = \partial_\sigma(\epsilon^\sigma A{\cdot}J) + \left[\partial_\tau\epsilon_\mu + \partial_\mu\epsilon_\tau - \tfrac{1}{2}(\partial\cdot\epsilon)\eta_{\mu\tau}\right]A^\tau J^\mu \ ,$$

and hence the interaction term is (classically) invariant (up to total derivatives) under both scale transformations and special conformal transformations.

(d). Consider now the interaction term

$$\mathcal{L}_1 = A^\mu K_\mu \quad \text{where} \quad K_\mu = (\bar{\phi}\,\partial_\mu\phi + \phi\,\partial_\mu\bar{\phi}) \ .$$

Show that

$$\delta K_\mu = \tfrac{1}{2}(\partial\cdot\epsilon)K_\mu + \epsilon^\sigma\partial_\sigma K_\mu + K_\tau\partial_\mu\epsilon^\tau + \tfrac{1}{2}\phi\bar{\phi}\partial_\mu(\partial{\cdot}\epsilon) \ .$$

$$\delta\mathcal{L}_2 = \partial_\sigma(\epsilon^\sigma A{\cdot}K) + \left[\partial_\tau\epsilon_\mu + \partial_\mu\epsilon_\tau - \tfrac{1}{2}(\partial\cdot\epsilon)\eta_{\mu\tau}\right]A^\tau K^\mu + \tfrac{1}{2}\phi\bar{\phi}A^\sigma(\partial_\sigma\partial_\tau\epsilon^\tau) \ .$$

Hence, this term is invariant (up to total derivatives) for conformal transformations for which $\partial_\sigma\partial_\tau\epsilon^\tau = 0$, ie for translations, rotations, scale transformations but not special conformal transformations (explain why).

3. Specialisation to $d = 2$

3.1 Conformal transformations in two dimensions

Show that the equations

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \delta_{\mu\nu}(\delta^{\sigma\tau}\partial_\sigma \epsilon_\tau) \ ,$$

in flat two-dimensional Euclidean space in Cartesian coordinates are equivalent to the Cauchy Riemann equations

3.2 Special conformal transformations of the complex plane

Show that the special conformal transformation (1) gives

$$z \mapsto \frac{z}{1 - \bar{b}z} \ , \quad \bar{z} \mapsto \frac{\bar{z}}{1 - b\bar{z}} \ .$$

Consider the infinitesimal transformations

$$\delta z = b z^2 \ , \quad \delta \bar{z} = \bar{b}\bar{z}^2 \ ,$$

generated by

$$Q = bL_1 + \bar{b}\bar{L}_1 \ .$$

We can usually take the transformations of z and $\bar{z}$ to be independent, but if we require that they are complex conjugate, so that $b = (\alpha + i\beta)$ and $\bar{b} = (\alpha - i\beta)$ with real coefficients α and β, show that this leads to the following infinitesimal transformations of the Cartesian coordinates:

$$\delta x = \alpha x(x^2 - y^2) - 2\beta xy \ , \quad \delta y = 2\alpha xy + \beta(x^2 - y^2)$$

generated by

$$Q = \alpha(L_1 + \bar{L}_1) + \beta i(L_1 - \bar{L}_1) \ .$$

3.3 Möbius maps

Consider the map

$$z \mapsto \frac{az + b}{cz + d}$$

Show that there is a unique map which takes any three points to any other three points.

Verify that an infinitesimal Möbius map is of the required form.

3.4 Quasiprimary state

Show that if $|\psi\rangle$ is a quasi-primary state of weight h, then the new states $|\chi\rangle$ defined by

$$|\chi\rangle = (L_{-2} - \frac{3}{4h + 2} L_{-1}L_{-1})|\psi\rangle \ ,$$

is a quasi-primary state of weight $(h + 2)$.

3.5 One–, two– and three–point functions

In this question all dependence on anti-holomorphic coordinates $\bar{z}_i$ is suppressed. Recall that the vacuum state $|0\rangle$ obeys

$$0 = L_{-1}|0\rangle = L_0|0\rangle = L_1|0\rangle = \langle 0|L_{-1} = \langle 0|L_0 = \langle 0|L_1$$

and that a primary field satisfies

$$[L_m, \phi_h(z)] = z^m(h(m+1)\phi_h(z) + z\frac{\partial \phi_h}{\partial z}(z)) \ ,$$

(a) Use invariance of the vacuum to show that the expectation value of a single primary field vanishes if it has non-zero conformal weight i.e. that

$$0 = \langle 0|\phi_h(z)|0\rangle \quad \text{if } h \neq 0$$

(b) Use the invariance of the vacuum under the action of L_{-1} to show that the two-point function of two primary fields is translation invariant, i.e.

$$\langle 0|\phi_h(z)\phi_{h'}(w)|0\rangle = f(z-w)$$

Next, use the invariance of the vacuum under the action of L_0 to show that the two-point function of two primary fields has the correct scale behaviour, i.e.q

$$\langle 0|\phi_h(z)\phi_{h'}(w)|0\rangle = \text{constant}(z-w)^{-h-h'}$$

Finally, use the invariance of the vacuum under the action of L_1 to show that the two-point function of two primary fields vanishes unless they have the same conformal weight.

(c) (*Harder*) Repeat this analysis for the three point function of three primary fields

$$\langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)|0\rangle$$

to show that this has the general form

$$\text{constant}(z_1-z_2)^{h_3-h_2-h_1}(z_2-z_3)^{h_1-h_3-h_2}(z_3-z_1)^{h_2-h_1-h_3}$$

4. Full infinite symmetry

4.1 Conformal invariance in light-cone coordinates

Consider a classically conformally conformally invariant theory in 2-dimensional Minkowski space. Consider light-cone coordinates $x^+ = t + x, x^- = t - x$.

Find the metric $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$.

Show that $\partial^\mu T_{\mu\nu} = 0$ and $T^\mu{}_\mu = 0$ imply that

$$\partial_- T_{++} = \partial_+ T_{--} = T_{+-} = 0$$

Show that for any functions f and g,

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left(f(x^+)T_{++}+g(x^-)T_{--}\right)\,\mathrm{d}x=0$$

4.2 Highest weight states

Show that if a state ψ satisfies both $L_1\psi=0$ and $L_2\psi=0$ then it satisfies $L_m|\psi\rangle=0$ for all $m>0$.

5. Brute Force

5.1 Determinant

Verify that, when expanded out,

$$A_2\,(h-h_{11})(h-h_{12})(h-h_{21})$$

where

$$c=13-6t-6/t\ ,\quad h_{rs}=\frac{r^2-1}{4t}+\frac{(s^2-1)t}{4}-\frac{rs-1}{2}\ ,$$

is equal to the the determinant $\det(M_2)$ for some A_2.

What are the straight lines in the plot of vanishing curves (in the lecture at 0:50)? Why are these the only lines in the region $1<c<25$?

6. Differential equations

6.1 Singular vectors

Recall a highest weight state $|\psi\rangle$ is one for which $L_1|\psi\rangle=L_2|\psi\rangle=0$.
Find the value of c for which $(L_{-2}L_{-2}-(3/5)L_{-4})|0\rangle$ is a highest weight state.

7. Recursion relations

7.1

What does the recursive formula over poles in c tell you about the expansion of a Virasoro conformal block over the set of global conformal blocks?

What does this have to do with the fact that the same factor (1/10) was present in both the expansion of the function

$$\frac{1}{1-z}+\frac{1}{z}-1$$

over global conformal blocks (at 0:38 in the lecture) and over Virasoro blocks (at 0:41 in the lecture)?

8. AGT: an exact formula

8.1 Liouville theory

The standard parametrisation of c and h is

$$c = 1 + 6Q^2 \ , \quad h = \alpha(Q - \alpha) \ , \quad Q = b + 1/b \ .$$

Let b be real. Show that $c \geq 25$. Suppose that $\alpha = Q/2 + iP$ with P real. Show that h is real and $h \geq (c-1)/24$.

Let $b = i\sqrt{t}$ with t real. Show that $c \leq 1$. Suppose that $\alpha = Q/2 + iP$ with P real. Show that h is real and $h \geq (c-1)/24$.

What relation does this have to the picture of vanishing Kac determinants in lecture 5?

9. "Old" Conformal Bootstrap

Ising model

Using the fact that

$$|\psi\rangle = (L_{-2} - \frac{4}{3} L_{-1} L_{-1}) |\sigma\rangle = 0 \ ,$$

show that

$$\langle \sigma | \sigma(1) | \sigma \rangle = 0 \ .$$

(one could just use symmetry under $\sigma \to -\sigma$ but not this assumption is not necessary).

For which values of h is $\langle h | \sigma(1) | \sigma \rangle$ allowwed to be non-zero?

Lee-Yang

Calculate the structure constant $C_{\phi\phi\phi}$ in the Lee-Yang model using the knowledge that ϕ is the only non-trivial primary field and that conformal blocks are given as in the lecture.

10. "New" Conformal Bootstrap

How do you think numerical results obtained this way compare to other results in physics?

i.e. are strict numerical bounds obtained from inequalities "better" or "worse" than approximate results or results with statistical errors?