Recursion relations

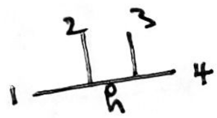
Idea: We know the behavior of

in some limit. It will have corrections.

Write as sum over corrections.

- Alternatively, analyse poles in expression for

For example:



$$= z^{h-h_3-h_4}\left\{ 1 + \frac{(h-h_1+h_2)(h+h_3-h_4)}{2h} z + \left[\frac{(\ )\, M_2^{-1}(\ )}{\det M_2}\right] z^2 + \dots \quad (\text{from } \S 5) \right\}$$

$$\begin{aligned}\det M_2 &= 2h\,(2h+1)(c+8h) - 36h^2 \\ &= 32\,(h-h_{11}(t))(h-h_{12}(t))(h-h_{21}(t)) \\ &= 2h\,(2h+1)(c-c^*) \qquad c^* = \frac{18h}{2h+1} - 8h\end{aligned}$$

Suppose $h = h_{12}(t) \Rightarrow t_{12}(h)$

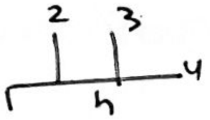
$$\begin{aligned}c_{12}(h) &= c(t_{12}(h)) = 13 - 6\,t_{12}(h) - \frac{6}{t_{12}(h)} \\ &= \frac{18h}{2h+1} - 8h\end{aligned}$$

Pole is at $c = c(t_{12}(h))$.

$$\begin{aligned}\det M_2 &= 2h\,(2h+1)(c - c(t_{12}(h))) \\ &= 2h\,(2h+1)(c - c_{12}).\end{aligned}$$

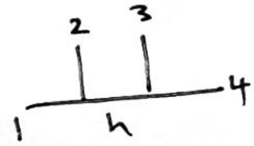
Large c behavior given by contribution from states which do not generate c in commutation relations.

Only $L_{-1}^{n}|h\rangle$ $\Rightarrow$ leading c behavior is



$$= z^{h-h_3-h_4}\underbrace{F(h-h_{12},\, h+h_{34},\, 2h,\, z)}_{\text{global block}} + \dots$$

We know



$$= z^{h-h_3-h_4}\Big\{ 1 + \frac{(h-h_{12})(h+h_{34})}{2h} z$$

$$+ z^2 \frac{(\ )\cdot M_2(\ )}{\det M_2} \qquad (\text{from } \S 5) \quad + \dots$$

(ii) What is remainder after taking off global block contribution

$$\frac{(h-h_{12})_2\,(h+h_{34})_2}{2!\,(2h)_2}\, z^2 \qquad ?$$

Answer: remarkably simple.

$$z^2 \cdot \frac{\left(h_1^2 - 3h_1^2 + h_2 + 6h_1h_2 - 3h_2^2 - h + 2h_1h + 2h_2h + h^2\right)\left(h_1 \to h_4,\ h_2 \to h_3\right)}{2\,(2h+1)^2\,(c - c_{12})}$$

These factors ensure the residue of the pole vanishes when the null state means the term doesn't actually appear.

Recall: If $h = h_{12}$, the state space is reduced
This factor is not present.

Check: factor vanishes if $(c = 1/2)$ $h = 1/2$, $h_1 = h_2 = 1/16$

Result:

$F(h-h_{12},\ h+h_{34}, h, z)$ gets correction

$$+\sum_{\substack{m\geq 1\\ n\geq 2}} \frac{R_{mn}}{c-c_{mn}}\, F(\underset{\substack{\uparrow\\ \text{value at}\\ \text{the pole}}}{c_{mn}},\ \{h_i\},\ \underbrace{h+mn}_{\substack{\text{weight of}\\ \text{vanishing intermediate}\\ \text{state}}},\ z)$$

$$R_{mn} = \underbrace{\underset{\substack{\uparrow\\ \text{factors that}\\ \text{ensure residue vanishes}}}{P_{mn}}}_{\text{calculated}} \times \underbrace{\underset{\substack{\uparrow\\ \text{overall } h\text{-dependent factor.}}}{A_{mn}}}_{\text{guessed.}}$$

Formula works.

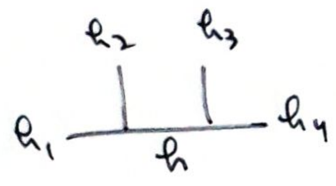
Can formally give a closed form expression for the recursion (Perlmutter).

2nd recursion: repeat for large h / poles in h

Leading behaviour: can read off from series expansion:

$$1+\frac{h}{2}x+\frac{h^2x^2}{8}+\frac{h^3x^3}{48}+\frac{h^4x^4}{384}\cdots = e^{xh/2}$$

Actually, much more complicated



$$\sim (16q)^h \left(\frac{z(1-z)}{16q}\right)^{\frac{c-1}{24}} \theta_3(q)^{\frac{c-1}{2}-4\sum h_i}$$

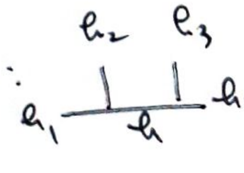
($z^{-h_3-h_4}$ missing..)

Here $\theta_3(q)=\sum_{n=-\infty}^{\infty} q^{n^2}$, $\quad q=e^{-\pi\frac{K(1-z)}{K(z)}}$

$$K(z)=\frac{\pi}{2}F\left(\tfrac{1}{2},\tfrac{1}{2};1;z\right)=\frac{1}{2}\int_0^1\frac{dt}{\sqrt{t(1-t)(1-zt)}}$$

$$=\frac{\pi}{2}\theta_3^2(q)$$

Recursion:



$$= \left[\;\cdots\cdots\;\right]\cdot H(c,\{h_i\},h,q)$$

$$H(c,\{h_i\},h,q)=1+\sum_{\substack{m,n\\ \geq 1}}\frac{(16q)^{mn}R_{mn}'}{h-h_{mn}(c)}\cdot H(c,\{h_i\},h_{mn}+mn,q)$$

(Again, can be resummed)

These are

- Very effective numerically
- somewhat tricky in cases when poles arise.
- Very effective at large c / large h.

Next: Exact formula.